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Un Nuevo Algoritmo Evolutivo Multi-Objetivo Basado en el Indicador $R2$

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Raquel Hernández Gómez

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Dr. Carlos Artemio Coello Coello

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A New Multi-Objective Evolutionary Algorithm Based on the $R2$ Indicator

Submitted by

Raquel Hernández Gómez

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Dr. Carlos Artemio Coello Coello

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Resumen

Una gran número de problemas del mundo real requieren de la optimización simultánea de varios objetivos en conflicto. A éstos se les conoce como problemas de optimización multi-objetivo y no tienen una solución única, sino un conjunto de soluciones que representan los diferentes compromisos entre los objetivos. Éste es el denominado conjunto de óptimos de Pareto y a su imagen se le llama, frente de Pareto.

El uso de algoritmos evolutivos y metaheurísticas bio-inspiradas para resolver problemas multi-objetivo se ha vuelto muy popular en los últimos 15 años, dando lugar a una amplia variedad de algoritmos evolutivos multi-objetivo (AEMOs). Los dos componentes principales de un AEMO son: (1) un mecanismo de selección que preserva los mejores compromisos entre los objetivos (i.e, soluciones no dominadas) y (2) un estimador de densidad que permite esparcir soluciones uniformemente a lo largo del frente de Pareto, de tal manera que sean lo más diversas posibles.

La incorporación de indicadores de desempeño como el mecanismo de selección en un AEMO es un tema que ha atraído mucho interés en años recientes. Esto debido a que los esquemas de selección basados en la optimalidad de Pareto no tienen un buen desempeño en problemas con cuatro o más funciones objetivo. El indicador que ha sido más utilizado para ser incorporado en el mecanismo de selección de un AEMO es el hipervolumen. Sin embargo, en el presente trabajo, exploramos el uso del indicador $R2$, que presenta varias ventajas con respecto al hipervolumen, de entre las cuales, la principal es su bajo costo computacional. En esta tesis, proponemos un nuevo AEMO, llamado MOMBI (Many-Objective Metaheuristic Based on the $R2$ Indicator), el cual jerarquiza individuos por medio de funciones de utilidad. El enfoque propuesto es comparado con NSGA-II (basado en dominancia de Pareto), MOEA/D (basado en escalarización), SMS-EMOA (basado en el hipervolumen), Δ_p-DDE (basado en el indicador Δ_p) y otros algoritmos de reciente creación basados en el indicador $R2$, utilizando diversos problemas de prueba estándar.

Nuestros resultados experimentales indican que MOMBI obtiene soluciones de calidad similar a los producidos por SMS-EMOA, pero a un menor costo computacional. Adicionalmente, MOMBI supera a MOEA/D y Δ_p-DDE en la mayoría de las instancias de prueba, particularmente cuando se trata de problemas de más de tres funciones objetivo con frentes de Pareto complicados. De tal forma, creemos que el enfoque propuesto es una alternativa viable para resolver problemas de optimización con varios objetivos.

Abstract

A wide variety of real-world problems have several (often conflicting) objectives that need to be optimized at the same time. They are called multi-objective optimization problems (MOPs) and their solution involves finding a set of decision variables, also known as Pareto optimal set, that represent the best trade-offs among all the objectives. The image of the Pareto optimal set is called the Pareto optimal front.

The use of evolutionary algorithms, as well as other bio-inspired metaheuristics, for solving MOPs has become increasingly popular in the last 15 years, giving rise to a wide variety of multi-objective evolutionary algorithms (MOEAs). The two key algorithmic components of a MOEA are: (1) a selection mechanism that preserves the best possible trade-offs among the objectives (i.e., the so-called nondominated solutions) and (2) a density estimator that allows us to spread solutions along the Pareto optimal front in a uniform way, so that they are as diverse as possible.

The incorporation of performance indicators as the selection mechanism of a MOEA is a topic that has attracted increasing interest in the last few years. This has been mainly motivated by the fact that Pareto-based selection schemes do not perform properly when solving problems with four or more objectives. The indicator that has been most commonly used for being incorporated in the selection mechanism of a MOEA has been the hypervolume. Here, however, we explore the use of the $R2$ indicator, which presents some advantages with respect to the hypervolume, the main one being its low computational cost. In this document, we propose a new MOEA called Many-Objective Metaheuristic Based on the $R2$ Indicator (MOMBI), which ranks individuals using utility functions. The proposed approach is compared with respect to NSGA-II (based on Pareto dominance), MOEA/D (based on scalarization), SMS-EMOA (based on hypervolume), Δ_p-DDE (based on Δ_p indicator), and some other recently created MOEAs based on the $R2$ indicator, using several benchmark problems.

Our preliminary experimental results indicate that MOMBI obtains solutions of similar quality to those produced by SMS-EMOA, but at a much lower computational cost. Additionally, MOMBI outperforms MOEA/D and Δ_p-DDE in most of the test instances adopted, particularly when dealing with high-dimensional problems having complicated Pareto optimal fronts. Thus, we believe that our proposed approach is a viable alternative for solving many-objective optimization problems.

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Chapter 1

Introduction

Multi-objective optimization problems are present in many areas of our life. Typical application areas are engineering (robotics, transportation, aeronautics, telecommunications, etc.), chemistry, medicine, ecology, and finance, just to name a few. These problems are hard to solve, and even a slight improvement in a solution often has a direct impact on costs and other important factors such as time, profit and customer satisfaction. Therefore, the presence of several objectives in a problem, gives rise to a set of trade-off solutions (instead of a single one), which become incomparable in the absence of any further information. This makes necessary to find as many solutions as possible.

Multi-objective Evolutionary Algorithms (MOEAs) are a specific class of solvers of multi-objective problems, which simulate the natural evolutionary process, based on the Darwinian concept of "survival of the fittest". They apply a set of variation operators, such as crossover and mutation, to combine different solutions, aiming to produce reasonably good approximations of Pareto optimal sets.

MOEAs based on Pareto dominance have been used successfully in problems for two or three objectives. However, they do not perform properly when dealing with more than three objectives. In this thesis, we want to address this issue by integrating into a MOEA a performance indicator of low computational cost.

The organization of this chapter is as follows: Section 1.1 presents the background and motivation of this work. Section 1.2 defines the problem statement and our proposed approach. Section 1.4 briefly discusses some previous work. Finally, Section 1.5 provides the organization of this thesis.

1.1 Background and Motivation

For many years, MOEAs have adopted selection mechanisms based on Pareto optimality. These mechanisms preserve solutions that are Pareto optimal with respect to a set of reference (normally the current population), and assign a rank to each of these solutions, such that all the nondominated solutions are considered to be equally good. Pareto-based MOEAs have been very popular since the 1990s, but recent studies have shown that they do not perform properly when dealing with problems having four or more objectives.[1] This has motivated the development of new selection schemes from which the use of quality assessment indicators is probably the most popular [3]. The idea in this case, is to optimize a quality assessment indicator that provides a good ordering among sets that represent Pareto approximations.

From the many indicators currently available, the hypervolume[2] [5] is, with no doubt, the most popular. The main advantage of the hypervolume indicator is that it has been proved that its maximization is equivalent to finding the Pareto optimal set [6], and this has also been empirically corroborated [7]. In fact, maximizing the hypervolume also leads to sets of solutions whose spread along the Pareto optimal front is maximized (although this does not necessarily mean that such solutions will be uniformly distributed along the Pareto optimal front). Nevertheless, the high computational cost of the hypervolume (its computational cost grows exponentially on the number of objectives [8]) normally makes a selection mechanism based on such indicator prohibitive for problems having more than 5 objectives [9]. The nice mathematical properties of the hypervolume indicator has triggered an important amount of research, including work that focuses on computing it in a more efficient way [10, 11]. It is indeed possible to approximate the hypervolume contribution, significantly reducing its computational cost [10], but few studies of the performance of such approaches with respect to those using exact hypervolume calculations are currently available.

1.2 Problem Statement and Proposal

The main goal of this thesis is to incorporate a performance indicator of low computational cost that works as a selection mechanism in an evolutionary algorithm, in order to solve unconstrained multi-objective problems.

In this work, we explore the use of another indicator that is known to have nice mathematical properties [12]: $R2$. Here, we propose a new MOEA, called Many-Objective Metaheuristic Based on the $R2$ Indicator (MOMBI) and analyze its performance with respect to that of some well-known approaches: NSGA-II [13], which is based on Pareto dominance, MOEA/D [14], based on scalarization, SMS-EMOA [3], based on the hypervolume indicator (we use the approach to approximate the hypervolume contribution proposed in [10]), Δ_p-DDE, based on Δ_p indicator, and new MOEAs based on the $R2$ indicator: $R2$-IBEA [15], $R2$-MODE [16] and $R2$-MOGA [16].

Even though, we are dealing with stochastic algorithms, the methodology adopted for the analysis of MOEAs relies on the scientific method.

[1] Also called many-objective optimization problems, and its derived area, many-objective optimization, is currently considered a hot research topic [1, 2].

[2] The **hypervolume** (also known as the S metric or the Lebesgue Measure) of a set of solutions measures the size of the portion of objective space that is dominated by those solutions collectively [4].

1.3 Contributions

The main contributions of this work can be summarized as follows:

- A new MOEA that incorporates a ranking algorithm of individuals based on the $R2$ indicator.
- A weight vector generation[3] based on Tabu Search and Uniform Design.
- An interactive MOEA, in which the preferences of the user can be integrated.
- A detailed comparative study of the proposal with other state-of-the-art MOEAs.
- An analysis of sensitivity of the proposal to the parameters specific to the $R2$ indicator.
- Complexity analysis [17] of the compared MOEAs.
- A peer-reviewed paper presented at CEC'2013 [18].

1.4 Previous Work

In this section we review the previous related work on the use of indicators in the selection mechanism of a MOEA.

As indicated before, the performance indicator that has been most commonly used for the selection mechanism of a MOEA is the hypervolume [4]. This indicator has several advantages, from which the main one is that it is the only unary indicator which is known to be strictly monotonic [19]. However, computing the hypervolume is exponential in the number of objectives [20] and is sensitive to the choice of the reference point [3].

Currently, there are several MOEAs that incorporate the hypervolume in their selection mechanism (e.g., the S Metric Selection-Evolutionary Multi-Objective Optimization Algorithm (SMS-EMOA) [3] and the multi-objective covariance matrix adaptation evolution strategy (MO-CMA-ES) [21]). However, the high computational overload of these approaches motivated the development of alternative strategies. One of them is to estimate (by means of Monte Carlo simulations) the ranking of a set of individuals that would be induced by the hypervolume indicator, without having to compute the exact indicator values. This is the approach adopted by the Hypervolume Estimation algorithm for multi-objective optimization (HypE) [10].

More recently, a new performance indicator called Δ_p was proposed in [22]. This indicator can be seen as an "averaged Hausdorff distance" between the outcome set

[3]The $R2$ indicator preserves diversity by means of a set of weight vectors that must be uniformly distributed across the objective space.

and the Pareto optimal front. Δ_p is composed of slight modifications of two well-known performance indicators: generational distance (see [23]) and inverted generational distance (see [24]). Δ_p is a pseudo-metric which simultaneously evaluates proximity to the Pareto optimal front and spread of solutions along it. Although Δ_p is not Pareto compliant, its computation has a much lower computational cost than that of the hypervolume, and it can also handle outliers, which makes it attractive for assessing performance of MOEAs. It is worth noting, however, that for incorporating Δ_p into the selection mechanism of a MOEA, it is necessary to have an approximation of the true Pareto optimal front at all times. This has motivated the development of techniques that can produce such an approximation in an efficient and effective way. For example, in [25], the authors linearize the nondominated (piecewise linear) front of the current population, and include this mechanism in the Δ_p-EMOA, which is used for solving bi-objective optimization problems. This algorithm is inspired by SMS-EMOA, and is assisted by a secondary population. Δ_p-EMOA performs better than NSGA-II [13], while consuming a lower number of function evaluations. An extension of this approach to three-objective problems is reported in [26]. In this case, the algorithm requires some previous mathematical steps which include reducing the dimensionality of the nondominated solutions and calculating their convex hull. This version of Δ_p-EMOA achieves a better distribution of solutions than MOEA/D [14], SMS-EMOA and NSGA-II. However, this MOEA requires additional parameters and consumes a high computational time when dealing with many-objective optimization problems.

Another possible approach to incorporate Δ_p into a MOEA is to use an echelon form of the nondominated individuals for the Pareto optimal front. This is the mechanism adopted in Δ_p-DDE [9], in which Δ_p is used as the selection mechanism of a differential evolution algorithm. Δ_p-DDE was able to outperform NSGA-II and provided competitive results with respect to SMS-EMOA, but at a considerably lower computational cost for many-objective optimization problems. The main limitation of this approach is that it produces a poor spread of solutions in high-dimensional search spaces. Also, it has some difficulties for dealing with discontinuous Pareto optimal fronts.

Recently, some researchers have recommended to adopt the $R2$ indicator proposed in [27] to compare approximation sets on the basis of a set of utility functions [12]. A utility function is a model of the decision maker's preference that maps each point in the objective space into a utility value. It is worth noticing that the $R2$ indicator is weakly monotonic, and it is correlated with the hypervolume but has a lower computational overhead than such indicator [12]. Because of this, the $R2$ indicator is widely recommended for dealing with many-objective optimization problems and over large nondominated sets [28]. It is worth emphasizing, however, that the main caveat when trying to use this performance indicator is that each utility function adopted, must be properly scaled.

The $R2$ indicator has been scarcely studied in the context of MOEAs. Here, we explore its potential use as a selection mechanism within a MOEA, emphasizing its possible usefulness in many-objective optimization problems.

1.5 Structure of the Thesis

Including this introduction, the thesis consists of six chapters and three appendixes.

Chapter 2 provides some basic concepts and definitions which are required as a background for the following chapters of this document.

Chapter 3 introduces some performance indicators that are used as selection mechanisms. A short review of the most representative MOEAs, that incorporate such indicators is also provided.

Our proposed approach is described in detail in Chapter 4.

The results obtained by the proposal are compared with respect to those generated by some state-of-the-art MOEAs, using standard test problems and performance indicators taken from the specialized literature in Chapter 5.

The conclusion of the thesis, as well as some possible paths for future research are presented in Chapter 6.

Appendix A presents the standard test problems used in the comparative study.

Appendix B introduces the methodology adopted for the analysis of MOEAs.

Finally, Appendix C presents a summary of the numerical results obtained in the experiments.

The implementation of the proposal, as well as the complete study can be downloaded at: `http://computacion.cs.cinvestav.mx/~rhernandez/mombi`.

Chapter 2

Preliminary Concepts

Optimization is the process of selecting the best candidate solution from a range of possibilities [29]. It aims for efficient allocation of scarce resources.

Many fields involve multiple conflicting objectives that should be optimized simultaneously. Such problems are generically known as Multi-Objective Optimization Problems (MOPs), and the area of multi-objective optimization, also known as multi-criteria optimization, is concerned with the solution of such problems.

In this chapter, some fundamental concepts of multi-objective optimization are introduced, emphasizing the motivation and advantages of using evolutionary algorithms in this domain. The rest of this chapter is organized as follows. We begin in Section 2.1 by introducing the necessary notation and formally describing MOPs. In Section 2.2, we provide the important notion of Pareto optimality and dominance. In Section 2.3, we define special points which are often used as reference solutions. In Section 2.4, we mention the role of the decision maker. In Section 2.5, we describe different features of MOPs, such as, multimodality, deceptiveness, bias, etc. We also point out different Pareto optimal front geometries, such as: linear, convex, concave, degenerate, disconnected, etc. In Section 2.6, we present several techniques for solving multi-objective problems, and we also state an important theorem that assures there is no universal robust solution technique for all MOPs. In Section 2.7, we focus on evolutionary algorithms, which are suitable for solving multi-objective problems, since they find multiple solutions in one single run. In Section 2.8, we describe a relevant area called many-objective optimization, which deals with problems having four or more objectives. Moreover, in Appendix A, we present benchmark test problems, which are designed for many-objective optimization and include many of the features described in Section 2.5. Finally, we conclude in Section 2.9 with a summary of the chapter.

This chapter is based on the introduction to multi-objective optimization given in [30], [31], and [32].

2.1 Multi-objective Optimization

Even though some real-world problems can be treated as single objective problems, it is hard, in general, to define all aspects of a problem in terms of only one objective. Defining multiple objectives often gives a better idea of the task. Multi-objective optimization, also known as multi-criteria optimization, has been available for about three decades, and its application in real-world problems is continuously increasing [33].

In single-objective optimization, the search space is often well defined. However, when there are several possibly contradicting objectives to be optimized simultaneously, there is no longer a single optimal solution but rather a whole set of possible solutions of equivalent quality. When we try to optimize several objectives at the same time, the search space also becomes partially ordered. To obtain the optimal solutions, there will be a set of optimal trade-offs among the conflicting objectives. Moreover, such objectives may also be *incommensurable* (i.e., measured in different units).

A Multi-Objective Optimization Problem (MOP) can be written in the form:

$$\text{Minimize} \quad \{f_1(\vec{x}), f_2(\vec{x}), \ldots, f_m(\vec{x})\} \tag{2.1}$$

$$\text{subject to} \quad g_i(\vec{x}) \geq 0 \quad i = 1, 2, \ldots, p \tag{2.2}$$

$$h_i(\vec{x}) = 0 \quad i = 1, 2, \ldots, q, \tag{2.3}$$

where we have m (≥ 2) *objective functions* $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$. We denote the vector of objective functions by $\vec{f}(\vec{x}) := (f_1(\vec{x}), f_2(\vec{x}), \ldots, f_m(\vec{x}))^T$. The *decision variable vector* $\vec{x} = (x_1, x_2, \ldots, x_n)^T$ belongs to the *feasible region set* $\mathcal{S}$, which is a subset of the *decision variable space* $\mathbb{R}^n$, and it is defined by the constraint functions $g_i, h_j : \mathbb{R}^n \rightarrow \mathbb{R}$ for all $i = 1, \ldots, p$, $j = 1, ..., q$.

In other words, it is aimed to determine from among the set of all numbers which satisfy equations (2.2) and (2.3) the particular set $x_1^*, x_2^*, \ldots, x_n^*$ which yields the optimum values of all the objective functions.

The above-described constraints represent restrictions imposed by a particular characteristic of the environment or resources available (e.g., physical limitations, time restrictions, etc.). It is worth noting that q, the number of *equality constraints*, must be less than n, the number of decision variables, because if $q \geq n$ the problem is said to be *overconstrained*, since there are more unknowns than equations.

We denote the image of the feasible region by $\mathcal{Z} := \vec{f}(\mathcal{S})$ and call it a *feasible objective region*. It is a subset of the *objective space* $\mathbb{R}^m$.

For simplicity, we assume that all the objective functions are to be minimized. The following identity may be used to transform all the functions which are to be maximized into a form which allows their minimization:

$$\max f_i(\vec{x}) := \min\left(-f_i(\vec{x})\right). \tag{2.4}$$

If we want to calculate the optimal function values from the original objective functions, we need to multiplying the transformed functions by -1. Similarly, inequality constraints of the form:

$$g_i(\vec{x}) \leq 0 \quad i = 1, 2, \ldots, p \tag{2.5}$$

can be transformed to (2.2) form by multiplying by -1 and changing the sign of the inequality. Thus, equation (2.5) is equivalent to

$$-g_i(\vec{x}) \geq 0 \quad i = 1, 2, \ldots, p \tag{2.6}$$

The notion of optimality changes when having several objective functions, because in MOPs, the aim is to find good compromises rather than a single solution as in global optimization. In the next section, we discuss this in more detail.

2.2 Pareto Optimality and Dominance

Multi-objective optimization problems are in a sense ill-defined, since there is no natural ordering in the objective space. For example $(2, 2)^T$ can be said to be less than $(5, 5)^T$, but how to compare $(2, 0.7)^T$ and $(0.7, 2)^T$.

Anyway, some of the objective vectors can be extracted for examination. Such vectors are those where none of the components can be improved without deterioration of at least one of the other components. Francis Ysidro Edgeworth [34] presented this definition in 1881. However, the definition is usually called Pareto optimality after the French-Italian economist and sociologist Vilfredo Pareto, who, in 1896, developed it further (see [35, 36]). However, some authors, like Stadler [37], use the term Edgeworth-Pareto optimality for recognizing the true origin of this definition. Koopmans was one of the first to employ in 1951 the concept of Pareto optimality in [38]. A more formal definition of Pareto optimality is the following:

Definition 2.2.1. A decision vector $\vec{x} \in \mathcal{S}$ is *Pareto optimal* if there does not exist another decision vector $\vec{y} \in \mathcal{S}$ such that $f_i(\vec{y}) \leq f_i(\vec{x})$ for all $i = 1, \ldots, m$ and $f_j(\vec{y}) < f_j(\vec{x})$ for at least one index j.

In other words, this definition says that $\vec{x}$ is Pareto optimal if there exists no feasible vector $\vec{y}$ which would decrease some criterion without causing a simultaneous increase in at least one other criterion.

Therefore, in a MOP, our aim is to determine the Pareto optimal set from the set $\mathcal{S}$ of all the decision variable vectors that satisfy (2.2) and (2.3). Note however that in practice, not all the Pareto optimal set is normally desirable (e.g., it may not be desirable to have different solutions that map to the same values in objective function space) or achievable.

Other important definitions associated with Pareto optimality for a given MOP $\vec{f}(\vec{x})$ are the following:

Definition 2.2.2. The *Pareto Optimal Set* $\mathcal{P}^*$ is defined by:

$$\mathcal{P}^* := \{\vec{x} \in \mathcal{S} \mid \vec{x} \text{ is Pareto optimal}\}.$$

Definition 2.2.3. The *Pareto Optimal Front* $\mathcal{PF}^*$ is defined by:

$$\mathcal{PF}^* := \{\vec{f}(\vec{x}) \in \mathbb{R}^m \mid \vec{x} \in \mathcal{P}^*\}.$$

In addition to Pareto optimality, we define various preference relations associated to it. Let us assume that $\vec{x}$ and $\vec{y}$ are two given vectors in $\mathbb{R}^m$.

Definition 2.2.4 (Weak-Pareto Dominance)**.** A solution $\vec{x}$ is said to *weakly dominate* a solution $\vec{y}$ (denoted by $\vec{x} \preceq \vec{y}$), if and only if $\forall i \in \{1, \ldots, m\}$, $x_i \leq y_i$.

Definition 2.2.5 (Pareto Dominance)**.** A solution $\vec{x}$ is said to *dominate* a solution $\vec{y}$ (denoted by $\vec{x} \prec \vec{y}$), if and only if $\vec{x}$ is partially less than $\vec{y}$; i.e., $\forall i \in \{1, \ldots, m\}$, $x_i \leq y_i \wedge \exists j \in \{1, \ldots, m\}$ such that $x_j < y_j$. This relation may also be stated in terms of the weak-Pareto dominance, $\vec{x} \prec \vec{y} := (\vec{x} \preceq \vec{y}) \wedge (\vec{x} \neq \vec{y})$.

Definition 2.2.6 (Strong-Pareto Dominance)**.** A solution $\vec{x}$ is said to *strongly dominate* a solution $\vec{y}$ (denoted by $\vec{x} \prec_S \vec{y}$), if and only if $\forall i \in \{1, \ldots, m\}$, $x_i < y_i$.

For the following definitions, let us assume that the symbol $\lhd$ can be any element of the set $\{\preceq, \prec, \prec_S\}$.

Definition 2.2.7. Vectors $\vec{x}$ and $\vec{y}$ are *incomparable* (denoted by $\vec{x} \parallel_{\lhd} \vec{y}$) with respect to $\lhd$, if and only if $\vec{x} \ntriangleleft \vec{y} \wedge \vec{y} \ntriangleleft \vec{x}$.

Definition 2.2.8. Vectors $\vec{x}$ and $\vec{y}$ are *indifferent* (denoted by $\vec{x} \sim \vec{y}$), if and only if $\forall i \in \{1, \ldots, m\}$, $x_i = y_i$.

Similar to the concept of Pareto optimality, weak and strong Pareto optimality can be derived [30]. The relationship between these concepts is that the strong-Pareto optimal set is a subset of the Pareto optimal set, which is a subset of the weak-Pareto optimal set.

Some authors [32] consider that weakly Pareto optimal solutions are not always useful in practice, because of the large size of the weakly Pareto optimal set. However, they are often relevant from a technical point of view because they are sometimes easier to generate than Pareto optimal points. Nevertheless, the most widely used relation is Pareto dominance.

Figure 2.1 illustrates the mapping between decision and objective spaces. The Pareto optimal set/front is represented by a fat line, while the weakly Pareto optimal set/front is represented by the fat and disconnected lines.

The following concepts express the general form of definitions 2.2.1 and 2.2.2 in terms of any of the previous relations.

Definition 2.2.9. A vector of decision variables $\vec{x} \in X$ is *nondominated* with respect to the set $X \subseteq \mathcal{S}$ and $\lhd$, if there does not exist another vector $\vec{y} \in X$ such that $\vec{f}(\vec{y}) \lhd \vec{f}(\vec{x})$.

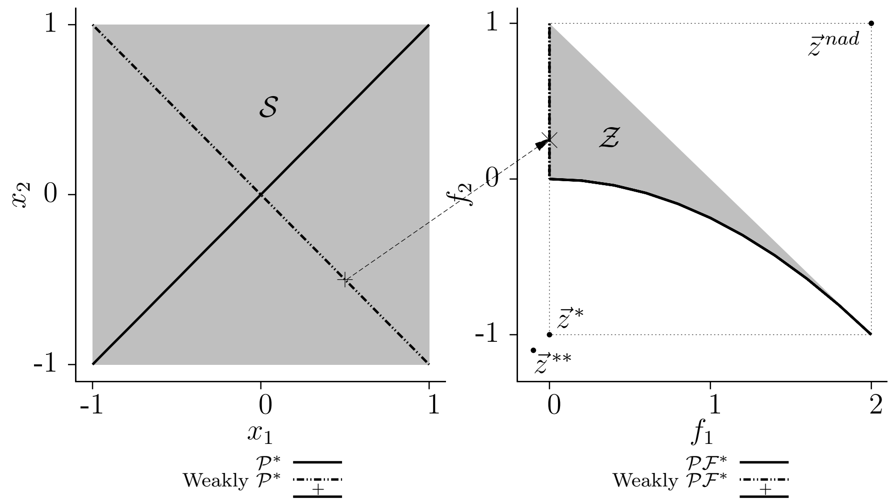


Figure 2.1: Mapping between spaces in $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ for the minimization problem where the decision space (left plot) consists of all pairs $(x_1, x_2) \in \{0, \ldots, 1\} \times \{0, \ldots, 1\}$ and the objective space (right plot) is defined by the two functions $f_1(x_1, x_2) = |x_1 + x_2|$ and $f_2(x_1, x_2) = -x_1 x_2$. The Pareto optimal set is $\{(x_1, x_2) \mid x_1 = x_2\}$, and the weakly Pareto optimal set is $\{(x_1, x_2) \mid x_1 = x_2, x_1 = -x_2\}$.

Definition 2.2.10. The set of *nondominated elements* relative to the set X and $\triangleleft$, is expressed by $ND(X, \triangleleft) := \{\vec{x} \in X \mid \vec{x} \text{ is nondominated}\}$.

The outcome of an optimizer is considered to be a set of mutually nondominated solutions also called *Pareto set approximation*.

With solution $\vec{a}$ as a point of reference, the regions in Figure 2.2 illustrate the different dominance relations. Solutions located in the second region are strongly dominated by solution $\vec{a}$ because $\vec{a}$ is better in both objectives. For the same reason, solutions located in the third region strongly dominate solution $\vec{a}$. Although $\vec{a}$ has a smaller objective value as compared to the solutions located at the boundaries between the first and second regions, and between the second and fourth regions, it only weakly dominates these solutions by virtue of the fact that they share a similar objective value along either one dimension. Solutions located in the first and fourth regions are incomparable to solution $\vec{a}$ because it is not possible to establish any superiority of one solution over the other solutions.

Zitzler et al. [39, 40] suggested three desirable aspects of nondominated sets:

1. **Convergence:** The distance of the resulting nondominated front to the Pareto optimal front should be minimized.

2. **Distribution:** A good (in most cases uniform) distribution of the solutions found is desirable.

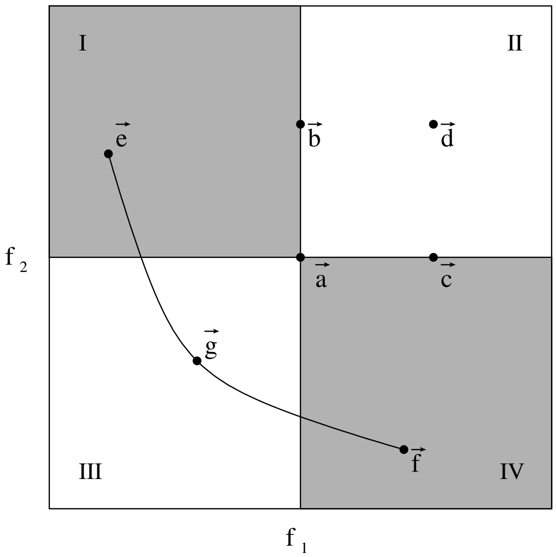


Figure 2.2: Examples of dominance relations. It holds that $\vec{a} \preceq \vec{a}$, $\vec{a} \preceq \vec{b}$, $\vec{a} \preceq \vec{c}$, $\vec{a} \preceq \vec{d}$, $\vec{a} \prec \vec{b}$, $\vec{a} \prec \vec{c}$, $\vec{a} \prec \vec{d}$, $\vec{a} \prec_S \vec{d}$, $\vec{a} \parallel \vec{e}$, $\vec{a} \parallel \vec{f}$, $\vec{a} \sim \vec{a}$. The nondominated front is $\left\{\vec{e}, \vec{g}, \vec{f}\right\}$.

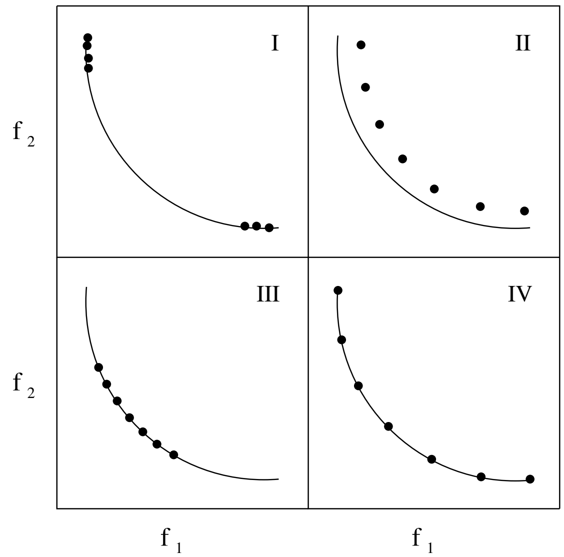


Figure 2.3: Desired features of approximation sets to the Pareto optimal front.

3. **Spread:** The extent of the obtained nondominated front should be maximized, i.e., for each objective, a wide range of values should be covered by the nondominated solutions.

In the literature, the term *proximity* is used instead of convergence, and the term *diversity* encompasses both distribution and spread. Good diversity is commonly of interest in objective-space, but may also be required in decision-space [41].

The above aspects are shown graphically in Figure 2.3. The first case is able to obtain solutions that are accurate and scattered; nonetheless, the set of solutions is not uniformly distributed. In the second case, a diverse set of well-spread solutions is obtained, although these are not accurate. The solutions in the third case are accurate and diverse; however, the edges of the Pareto optimal front are not explored. Finally, the fourth case shows the solution of an ideal optimizer.

In the remainder of this subsection, we describe the classifications of the relations $\preceq, \prec, \prec_S$, and $\parallel$. But before that, we provide some basic definitions of theory of relations [42, 43].

Definition 2.2.11. A *binary relation* $\mathcal{R}$ on a non-empty set X is defined to be a subset $\mathcal{R} \subseteq X \times X$. We write $x\mathcal{R}y$ if and only if $(x, y) \in \mathcal{R}$ and say that x is $\mathcal{R}$-related to y. We also write $x \not\mathcal{R}\, y$ where x is not $\mathcal{R}$-related to y.

Some important properties of binary relations are:

Reflexive: $\forall x \in X,\ x\mathcal{R}x$.

Irreflexive: $\forall x \in X,\ x \not\mathcal{R}\, x$.

Symmetric: $\forall x, y \in X,\ x\mathcal{R}y \Longrightarrow y\mathcal{R}x$.

Asymmetric: $\forall x, y \in X,\ x\mathcal{R}y \Longrightarrow y \not\mathcal{R}\, x$.

Antisymmetric: $\forall x, y \in X,\ (x\mathcal{R}y \wedge y\mathcal{R}x) \Longrightarrow x = y$, or equivalently, $(x\mathcal{R}y \wedge x \neq y) \Longrightarrow y \not\mathcal{R}\, x$.

Transitive: $\forall x, y, z \in X,\ (x\mathcal{R}y \wedge y\mathcal{R}z) \Longrightarrow x\mathcal{R}z$.

Definition 2.2.12. An *equivalence relation* is a binary relation over a set X which is reflexive, symmetric, and transitive.

Definition 2.2.13. A *partial order* is a binary relation over a set X which is reflexive, antisymmetric, and transitive.

Definition 2.2.14. If the partial order relation $\preceq$ is valid on a set X then the pair $(X, \preceq)$ is called *partially ordered set* (or *poset* for short).

Definition 2.2.15. A *strict partial order* is a binary relation over a set X which is irreflexive, asymmetric, and transitive.

Definition 2.2.16. A relation is asymmetric if and only if it is both antisymmetric and irreflexive.

Definition 2.2.17. Every asymmetric relation is also antisymmetric.

Definition 2.2.18. A transitive relation is irreflexive if and only if it is asymmetric.

Therefore, the antisymmetric property is satisfied for $\prec$ and $\prec_S$, because $x\mathcal{R}y$ and $y\mathcal{R}x$ are never both true.

In Table 2.1, we present the properties that meet each of the relations. We can observe that both strong Pareto dominance and Pareto dominance are strict partial orders, weak-Pareto dominance is a partial order and the indifference operator is an equivalence relation.

2.3 Reference Solutions

In this section, we define some special points which are often used as reference solutions in multi-objective optimization algorithms.

One of these points denotes the lower bounds of the Pareto optimal front for each objective function, which is expressed as follows:

Table 2.1: Properties of the different relations of dominance.

Relation	Property					
	Reflexive	Irreflexive	Symmetric	Asymmetric	Antisymmetric	Transitive
$\preceq$	✓				✓	✓
$\prec$ $\prec_S$		✓		✓	(✓)	✓
$\|_{\preceq}$		✓	✓			
$\|_{\prec}$ $\|_{\prec_S}$	✓		✓			
$\sim$	✓		✓			✓

Definition 2.3.1. Each element of the *ideal objective vector* $\vec{z}^* = (z_1^*, \ldots, z_m^*)$ minimizes each of the objective functions. The i-th component is defined as $z_i^* = \min_{\vec{x} \in \mathcal{S}} f_i(\vec{x})$.

It is obvious that this point corresponds to a non-existent solution, since there is some conflict among the objectives. In general, solutions closer to the ideal objective vector are better.

On the other hand, some algorithms may require a solution which has an objective value strictly better than all the solutions in the feasible objective region. For this purpose, the next reference point is defined as follows:

Definition 2.3.2. The *utopian objective vector* is an infeasible objective vector whose components are formed by $z_i^{**} = z_i^* - \epsilon_i$ for all $i = \{1, \ldots, m\}$, where ϵ_i is some small positive scalar.

It is worth noting that this vector strictly dominates every Pareto optimal solution.

Opposite to the ideal objective vector, the upper bounds of the Pareto optimal front are denoted in the next point:

Definition 2.3.3. The *nadir objective vector* $\vec{z}^{nad} = \left(z_1^{nad}, \ldots, z_m^{nad}\right)$ is constructed using the worst values of the objective functions in the complete Pareto optimal front $\mathcal{PF}^*$. Each i-th component is defined as $z_i^{nad} = \max_{\vec{x} \in \mathcal{P}^*} f_i(\vec{x})$.

The nadir objective vector may be feasible or not. In practice, it is usually difficult to obtain when having more than two objectives. Its components can be approximated using a pay-off table [32], which is formed by using the objective vectors obtained when calculating the ideal objective vector. That is, row i of the payoff table displays the values of all the objective functions calculated at the point where f_i obtained its minimal value. Hence, z_i^* is at the main diagonal of the table. The maximal value of the column i in the payoff table can be selected as an estimate of the upper bound of the objective f_i for $i = 1, ..., m$ over the Pareto optimal front.

However, this kind of an estimate is not necessarily too good. Some other methods have been proposed, see for example [32, 44].

Components of both the ideal and nadir objective vectors are useful for normalizing objective values in a common range. Figure 2.1 (right) shows these three reference points.

2.4 Decision Maker

Mathematically, every Pareto optimal point is an equally acceptable solution of the multi-objective optimization problem. However, it is generally desirable to obtain one point as a solution. Selecting one out of the set of Pareto optimal solutions calls for information that is not contained in the objective functions.

The person (or a group of persons) who is supposed to have better insight into the problem and who can express preference relations between different solutions is the *decision maker*.

Solving a multi-objective optimization problem calls for the co-operation of the decision maker and an analyst. By an *analyst* we here mean a person or a computer program responsible for the mathematical side of the solution process. The analyst generates information for the decision maker to consider and the solution is selected according to the preferences of the decision maker.

By solving a multi-objective optimization problem we here mean finding a feasible decision vector such that it is Pareto optimal and satisfies the needs and the requirements of the decision maker. Assuming such a solution exists, it is called a *final solution*. However, it may be difficult for the decision maker to distinguish between good and optimal solutions in real problems [45, 46]. If this is the case, the emphasis should be on finding good solutions.

We usually assume that decision makers are only interested in Pareto optimal points and the rest can be excluded. However, this is not the case if the problem has not been formulated well enough. Therefore, nondominated solutions may be important if there are some unformulated or hidden objective functions in the mind of the decision maker or some of the objective functions are simply proxies of the proper objective functions [45, 46]. In such cases, the Pareto optimal sets of the problem handled and the actual problem which should be solved, do not coincide. Here we assume the mathematical model to be accurate and static so that we can mainly concentrate on Pareto optimal solutions.

2.5 Features of Multi-Objective Problems

In the literature, the feasible region set $\mathcal{S}$ and the feasible objective region $\mathcal{Z}$ are better known as *search space* and *fitness space*, respectively. In this way, the Pareto optimal set is a subset of the search space, whereas the Pareto optimal front is a subset of the fitness space. The mapping from the search space to the fitness space defines the *fitness landscape*.

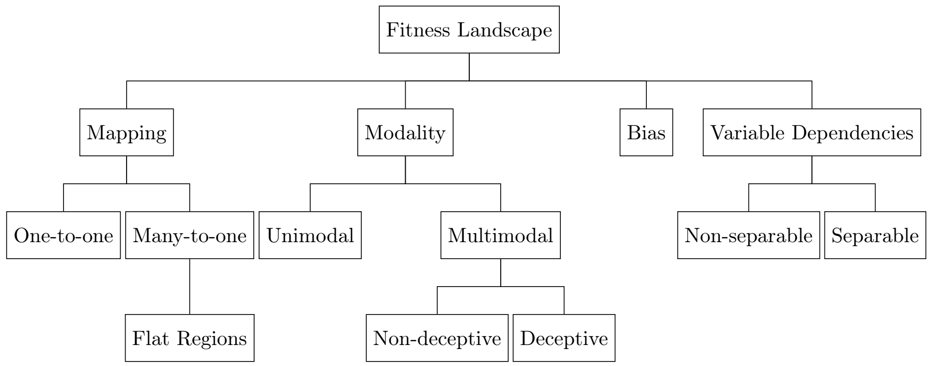


Figure 2.4: Features of the fitness landscape.

In this section, we are interested in describing the nature of the fitness landscape, and the relationship between the Pareto optimal set and the Pareto optimal front. The former identifies the types of difficulties encountered in the search space, whereas the latter influences our judgement of what is considered a "good" representative subset of the Pareto optimal set, which is important when it is impractical to identify the entire Pareto optimal set.

In the following two subsections, we present the features of the fitness landscape and the possible geometries that the Pareto optimal front can adopt. In some cases, these elements make real-world problems very difficult to solve for optimizers. For more details, the reader can consult [47, 48].

2.5.1 Fitness Landscape

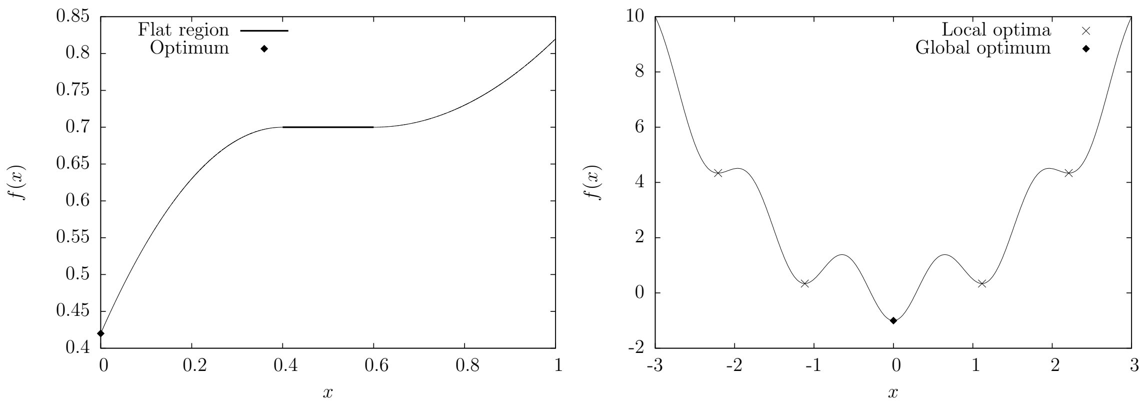
In Figure 2.4, we classify the features of the fitness landscape. Since such features are not exclusive, it is possible that a MOP may have a combination of them. Next, we describe each of these features.

The fitness landscape can be *one-to-one* or *many-to-one*. The many-to-one case presents more difficulties to the optimizer, as choices must be made between two decision vectors that evaluate to identical objective vectors. Likewise, the mapping between the Pareto optimal set and the Pareto optimal front may be one-to-one or many-to-one. In each case, we say that the problem is *Pareto one-to-one* or *Pareto many-to-one*, respectively.

A special instance of a many-to-one mapping occurs when a connected open subset of decision variable space maps to a singleton.[1] We refer to problems with this characteristic as problems with *flat regions*, that is, regions where small perturbations of the decision variables do not change the objective values. Optimizers can have difficulty with flat regions due to a lack of gradient information. In Figure 2.5(a), we illustrate this concept for a mono-objective problem.

If the majority of the fitness landscape is fairly flat, it does not provide useful

[1] A singleton, also known as a unit set [49], is a set with exactly one element.



(a) (b)

Figure 2.5: (a) Illustration of a flat function: $f(x) = 0.7 + 1.75 \min(0, x - 0.4)(0.4 - x) - 0.75 \min(0, 0.6 - x)(x - 0.6)$, where $x \in [0, 1]$. (b) Example of a multimodal function: $f(x) = x^2 + \sin\left(\frac{5}{3}\pi x - \frac{\pi}{2}\right)$, where $x \in [-3, 3]$.

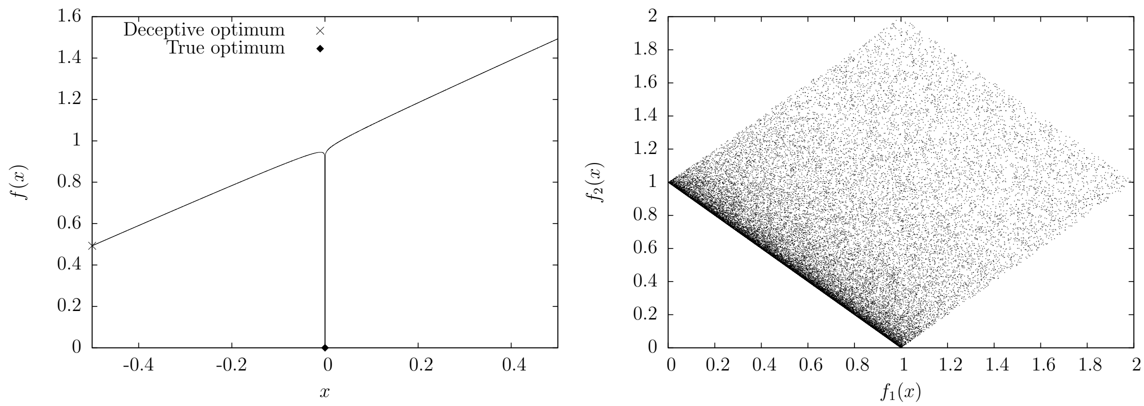
information regarding the location of Pareto optima, then the Pareto optima are said to be *isolated optima* [50]. Problems with isolated optima are very difficult to solve.

Another characteristic of fitness landscapes is *modality.* An objective function with only a single optimum is *unimodal.* An objective function is *multimodal* when it has multiple local optima. A *multimodal problem* is one that has a multimodal objective, see Figure 2.5(b) for an example.

A *deceptive objective function* has a special kind of multimodality. As defined by Deb [50], for an objective function to be deceptive it must have at least two optima, a true optimum and a deceptive optimum, but the majority of the search space must favor the deceptive optimum. A *deceptive problem* is one with a deceptive objective function. Multimodal problems are difficult because an optimizer can get stuck in local optima. Deceptive problems exacerbate this difficulty by placing the global optimum in an unlikely place. An example of this kind of problem is plotted in Figure 2.6(a).

Another characteristic of the fitness landscape is when an evenly distributed sample of decision vectors in the search space maps to an evenly distributed set of objective vectors in fitness space. We expect some variation in the distribution, but we are especially interested in significant variation, which is known as *bias.* Bias has a natural impact on the search process, particularly when the mapping from the Pareto optimal set to the Pareto optimal front is biased. Often, the decision maker has to choose between an even distribution of solutions with respect to the search space and with respect to the fitness space. The effects of bias are shown in Figure 2.6(b).

The judgement of whether a problem is biased on the density variation of solutions in decision variable space. While it is usually easy enough to decide whether a problem has bias or not, at the present time there is no agreed mathematical definition of bias (but see [51] for one possibility). Bias is perhaps best indicated by plotting solutions



(a) (b)

Figure 2.6: (a) Example of a deceptive function: $f(x) = x + |x|^{0.01}$, where $x \in [-0.5, 0.5]$. (b) The objective vectors that correspond to 40,000 randomly selected decision vectors from the biased two objective problem $f_1(x, y) = x^5 + y$, and $f_2(x, y) = x^5 + 1 - y$, where $x, y \in [0, 1]$. Note how, the objective vectors are denser toward $\mathcal{PF}^*$.

in fitness space.

Variable dependencies are an important aspect of a problem. Given a single objective O, a decision vector $\vec{x}$, and an index i, we define a derived problem $P_{O,\vec{x},i}$ as the problem of optimizing O by varying only x_i. This is a single objective problem with a single variable. We also define $P^*_{O,\vec{x},i}$ to be the set of global optima (in decision variable space) for each subproblem. If $P^*_{O,\vec{x},i}$, is the same for all values of $\vec{x}$, we say that x_i is *separable* on O. Otherwise, x_i is *nonseparable* on O.

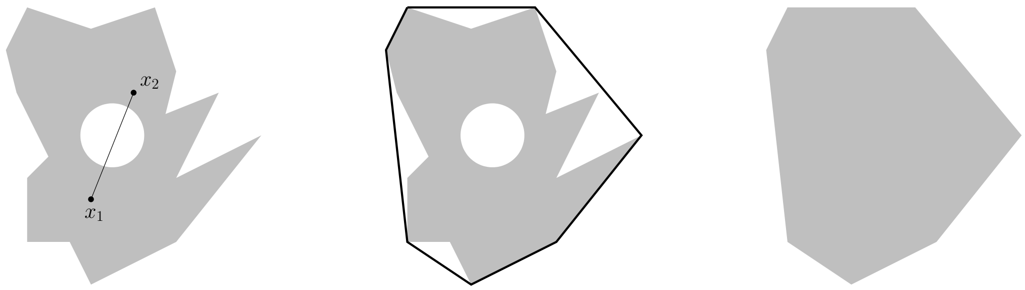
If every variable of O is separable, then O is a *separable objective.* Otherwise, O is a *nonseparable objective.* Similarly, if every objective of a problem P is separable, then P is a *separable problem.* Otherwise, P is a nonseparable problem. Separable problems can be optimized by considering each parameter in turn, independently of one another. A nonseparable problem is thus characterized by variable dependencies, and it is more difficult, and is more representative of real world problems.

2.5.2 Geometries of the Pareto Optimal Front

Unlike single objective problems, for which the Pareto optimal front is but a single point, Pareto optimal fronts for multi-objective problems can have a wide variety of geometries.

In this subsection we present the basic forms that Pareto optimal fronts can adopt, but before that, we review some related concepts [52, 53, 54].

Definition 2.5.1 (Convex Set)**.** A set X is *convex* if the line segment between any two points in X lies in X, i.e., if for any $\vec{x_1}, \vec{x_2} \in X$ and any λ with $0 \leq \lambda \leq 1$, we have:



(a) (b) (c)

Figure 2.7: A convex hull: (a) non-convex figure; (b) the boundary of the convex hull (the heavy line); (c) the convex hull.

$$\lambda\vec{x_1} + (1-\lambda)\vec{x_2} \in X. \tag{2.7}$$

Otherwise, the set is said to be *non-convex*.

Definition 2.5.2 (Convex Function)**.** A function $f : X \to \mathbb{R}$ is said to be *convex* at X if for any two points $\vec{x_1}, \vec{x_2} \in X$:

$$f(\lambda\vec{x_1} + (1-\lambda)\vec{x_2}) \leq \lambda f(\vec{x_1}) + (1-\lambda)f(\vec{x_2}), \tag{2.8}$$

where $0 \leq \lambda \leq 1$.

Definition 2.5.3 (Strictly Convex Function)**.** A function $f : X \to \mathbb{R}$ is said to be *strictly convex* at X if for any two other distinct points $\vec{x_1}, \vec{x_2} \in X$:

$$f(\lambda\vec{x_1} + (1-\lambda)\vec{x_2}) < \lambda f(\vec{x_1}) + (1-\lambda)f(\vec{x_2}), \tag{2.9}$$

where $0 < \lambda < 1$.

On the other hand, a function f is (strictly) concave if $-f$ is (strictly) convex.

Definition 2.5.4 (Convex Hull)**.** The *convex hull* of a set X, denoted Conv(X), is the set of all convex combinations of points in X:

$$\text{Conv}(X) = \left\{ \sum_{i=1}^{k} \lambda_i \vec{x_i} \mid \vec{x_i} \in X, (\forall i : \lambda_i \geq 0), \sum_{i=1}^{k} \lambda_i = 1 \right\}. \tag{2.10}$$

As the name suggests, the convex hull Conv(X) is always convex, and it is the smallest convex set that contains X. If a set X is non-convex, its convex hull is obtained by "filling in" all "non-convexities". Figure 2.7 illustrates the definition of convex hull.

Definition 2.5.5 (Neighborhood)**.** The *neighborhood* of a point $\vec{y}$ is the set of points:

$$A = \{\vec{a} \mid |\vec{a} - \vec{y}| < \epsilon\} \tag{2.11}$$

where ϵ is some small positive scalar.

Definition 2.5.6 (Interior Point)**.** A point $\vec{x} \in X$ is an *interior point* if there exists a neighborhood about $\vec{x}$ which contains only points of the set X.

Definition 2.5.7 (Boundary Point)**.** A point $\vec{x} \in X$ is a *boundary point* if an only if any neighborhood of $\vec{x}$ contains a point in X and a point not in X. The set of all boundary points in X is denoted by ∂X.

Definition 2.5.8 (Extreme Point)**.** An *extreme point* of a convex set is an element of the set which cannot be expressed as a convex combination of two other points in the set.

Obviously, an extreme point is a boundary point of the set, but all boundary points of a convex set are not necessarily extreme points; for example, the extreme points of a triangle are its vertices. The points which are not boundary points are interior points.

Another definition of a convex set in terms of its convex hull, is the following:

Definition 2.5.9 (Convex Set)**.** A set is convex if and only if it covers its convex hull.

Definition 2.5.10 (Concave Set)**.** A set is concave if and only if it is covered by its convex hull.

Definition 2.5.11 (Strictly Convex Set)**.** A set is *strictly convex* if and only if it is convex and all its boundary points are extreme points, or equivalently a set is strictly convex if it is convex and not concave.

Definition 2.5.12 (Strictly Concave Set)**.** A set is strictly concave if it is concave and not convex.

Definition 2.5.13 (Linear Set/Function)**.** A linear set or function is both convex and concave but neither strictly convex nor strictly concave.

Therefore, a MOP is convex (respectively concave or linear) if all objective functions are convex (respectively concave or linear) and the feasible region is convex (respectively concave or linear) [31].

Moreover, a *mixed front* is one with connected subsets that are each strictly convex, strictly concave, or linear, but not all of the same type.

A *degenerate front* is a front that is of lower dimension than the objective space in which it is embedded, less one. For example, a front that is a line segment in a three-objective problem is degenerate.

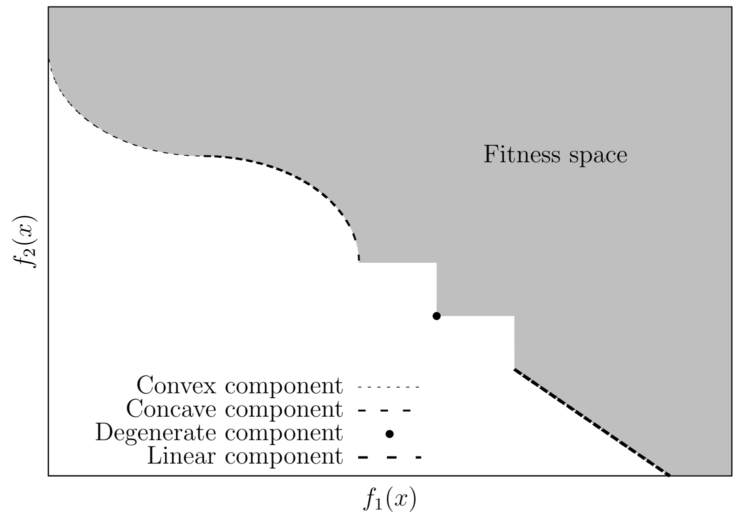


Figure 2.8: Sample geometry of a disconnected, mixed front that consists of a half-convex half-concave component, a degenerate point, and a linear component.

Degenerate Pareto optimal fronts can cause problems for some algorithms. For example, methods employed to encourage an even spread of solutions across the Pareto optimal front might operate differently should the front effectively employ fewer dimensions than expected.

Additionally, a front can be a *disconnected set*, often referred to as *discontinuous*. Pareto optimal sets can also be *disconnected*. Although disconnected Pareto optimal sets usually map to disconnected Pareto optimal fronts, this is not always the case. Figure 2.8 serves to clarify some of these geometries.

Another concept, commonly used in multi-objective optimization, is the knee:

Definition 2.5.14. The *knee* is a point on a (concave) convex Pareto optimal front that has the shortest distance to the (nadir) ideal objective vector.

Problems that are high-dimensional, discontinuous, multimodal, and/or *NP-Complete* [55, 56] are termed *irregular* [57]. In the following section we will review some methods that are often used to solve problems that exhibit these characteristics.

2.6 Methods for Solving Multi-Objective Problems

There are many methods available to tackle MOPs. According to Coello et al. [30], general search and optimization techniques are classified into three categories: enumerative, deterministic,[2] and stochastic (random). Many of these methods have been

[2] The term deterministic means that given a particular input, an algorithm will always produce the same output. Thus, no randomness is involved.

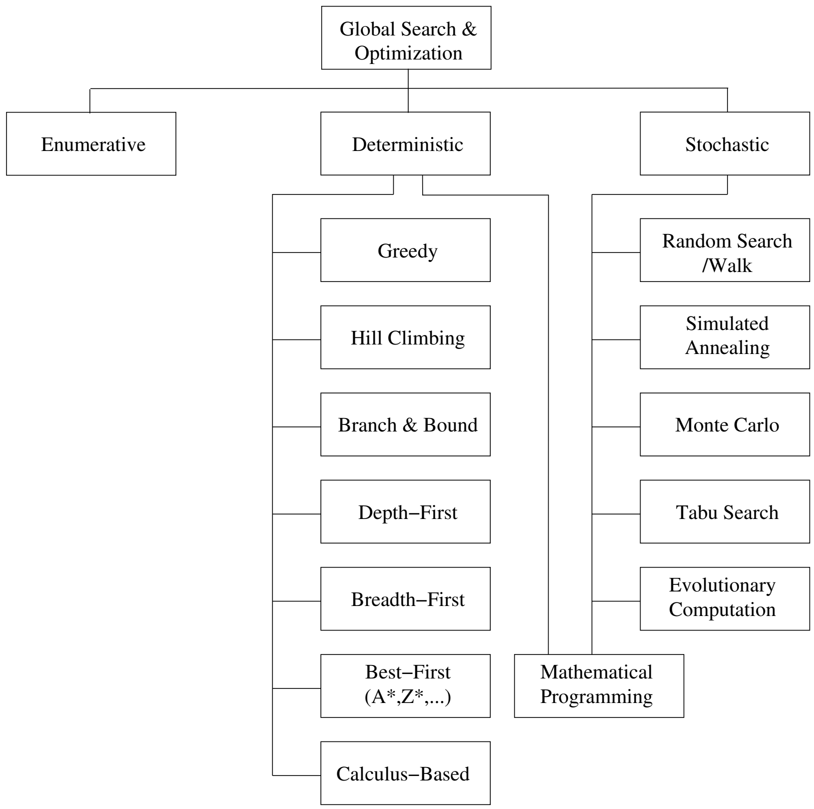


Figure 2.9: Global optimization approaches.

created primarily for single-objective optimization, and then their use has been extended to MOPs. Figure 2.9 shows common examples of each type. In the following paragraphs we briefly discuss each of them.

Enumerative schemes evaluate each possible solution within some defined finite search space, but they are inefficient or even infeasible as search spaces become large.

Deterministic algorithms limit the search space by incorporating knowledge about the problem domain, thus, they find "acceptable" solutions in "acceptable" time. These methods are successfully used when solving a wide variety of problems. However, they are often ineffective for problems which are irregular. The following heuristics[3] are within this classification:

- *Greedy algorithms* make locally optimal choices, assuming that optimal subsolutions are always part of the globally optimal solution. Thus, these algorithms fail unless that is the case.

- *Hill-climbing algorithms* proceed conservatively, changing only one solution feature at a time, and replacing the current best solution with the best one-feature change. These techniques work best on unimodal functions, but the presence

[3] A heuristic is a problem-solving technique in which the most appropriate local solution or partial solution is selected using comparative rules [30].

of local optima, plateaus, or ridges in the fitness (search) landscape reduce effectiveness.

- *Branch and bound search techniques* limit the search space, computing some bound at a given node which determines whether the node is "promising;" several nodes' bounds are then compared and the algorithm branches to the "most promising" node.

- *Depth-first search* is blind or uninformed in that the search order is independent of solution location. It expands a node, generates all successors, expands a successor, and so forth. If a node is determined "unpromising", backtracking to a node's parent is applied.

- *Breadth-first search* is also uninformed. It differs from depth-first search in its actions after node expansion, where it progressively explores the graph one layer at a time.

- *Best-first search* uses heuristic information to place numerical values on a node's "promise"; the node with highest promise is examined first.

- *Best-first search* selects a node to expand based both on "promise" and the overall cost to arrive at that node.

- *Calculus-based search methods* require continuity in some variable domain for an optimal value to be found.

Stochastic methods were developed as alternative approaches for solving irregular problems. They require a function assigning fitness values to possible (or partial) solutions, and an encode/decode (mapping) mechanism between the problem and algorithm domains. The majority of them cannot guarantee optimal solutions. However, in general they provide good solutions to a wide range of optimization problems which traditional deterministic search methods find difficult. Examples of these methods include the following metaheuristics[4]:

- A *random search* evaluates a given number of randomly selected solutions, while in a *random walk* the next solution is selected at random using the last evaluated solution as a starting point. Like enumeration, though, these strategies are not efficient for many MOPs because of their failure to incorporate problem domain knowledge. Random searches can generally expect to do no better than enumerative ones.

[4]The term *metaheuristic*, derives from the composition of two Greek words. *Heuristic* derives from the verb *heuriskein* which means "to find", while the suffix *meta* means "beyond, in an upper level". In general, metaheuristics are high level strategies which guide subordinate heuristics in the search process [58].

- *Simulated Annealing* is an algorithm modeled on an annealing analogy, where, for example, a liquid is heated and then gradually cooled down until it freezes. This algorithm consists in choosing a random move. If the move improves the current optimum, then it is always executed, else it is made with some probability $p < 1$. This probability exponentially decreases either by time or with the amount by which the current optimum is worsened. If water's temperature is lowered slowly enough it attains a lowest-energy configuration; the analogy is that if the "move" probability decreases slowly enough, the global optimum is found.

- *Monte Carlo methods* involve simulations dealing with stochastic events; they employ a pure random search where any selected trial solution is fully independent of any previous choice and its outcome. The current "best" solution and associated decision variables are stored as a comparator (these methods are not considered metaheuristics).

- *Tabu search* is a meta-strategy developed to avoid getting "stuck" in local optima. It keeps a record of both visited solutions and the "paths" which reached them in different "memories." This information restricts the choice of solutions to evaluate next. Tabu search is often integrated with other optimization methods.

- *Evolutionary Computation* is a generic term for several stochastic search methods which computationally simulate the natural evolutionary process. These techniques are loosely based on natural evolution and the Darwinian concept of "survival of the fittest" (see section 2.7 for more details).

Mathematical Programming involves several deterministic-search methods proposed by the *Operations Research* community. These methods are grouped with respect to the complexity of MOPs. Linear programming is designed to solve problems in which the objective function and all constraints are linear. Conversely, nonlinear programming techniques solve some MOPs not meeting those restrictions, but require convex constraint functions. Finally, stochastic programming is used when random-valued parameters and objective functions subject to statistical perturbations are part of the problem formulation. Depending on the type of variables used in the problem, several variants of these methods exist (i.e., discrete, integer, binary, and mixed-integer programming).

There are other techniques that can handle multiple objectives [30], such as Particle Swarm Optimization, Cultural Algorithms, Artificial Immune Systems, Cooperative Search, and Memetic Algorithms, just to mention a few.

One of the most widely used taxonomies for nonlinear programming is the one proposed by Hwang and Masud in 1979 [59, 32]. Here, the methods are categorized according to the participation of the decision maker in the solution process. The classes are:

- **No-preference Methods:** The opinions of the decision maker are not taken into consideration. The MOP is solved using some relatively simple method and the solution obtained is presented to the decision maker. The decision maker may either accept or reject the solution. These methods are suitable for situations where the decision maker does not have any special expectations of the solution and (s)he is satisfied simply with some optimal solutions.

- **A Posteriori Methods:** After the Pareto optimal set (or a part of it) has been generated, it is presented to the decision maker, who selects the most preferred among the alternatives. The inconveniences here are that the generation process is usually computationally expensive and, at least partially, difficult. On the other hand, it is hard for the decision maker to select from a large set of alternatives. A more important question is how to present or display the alternatives to the decision maker in an effective way.

- **A Priori Methods:** The decision maker must specify her or his preferences, hopes and opinions before the solution process takes place. The difficulty is that the decision maker does not necessarily know beforehand what it is possible to attain in the problem and how realistic her or his expectations are.

- **Interactive Methods:** Assuming the decision maker has enough time and capabilities for co-operation, these methods can be presumed to produce the most satisfactory results. Namely, only part of the Pareto optimal points has to be generated and evaluated, and the decision maker can specify and correct her or his preferences and selections as the solution process continues. This also means that it is not necessary to have any previous knowledge about the preference structure. However, the information should be meaningful and easy to understand.

These classical methods solve problems using mathematical techniques, such as gradient information. Therefore, some assumptions are made about continuity and convexity. Usually, they generate one solution, and the principal idea of these methods is to transform the multi-objective problem into a single objective problem using constraints, a hierarchy of objectives, or aggregation functions.[5] There is a large variety of classical methods. In Table 2.2, we present some of them.

No Free Lunch Theorem

As we have seen, there is a large variety of methods for solving MOPs. However, none of them can be said to be generally superior to all the others. This observation

[5] *Aggregation functions* combine all the objectives of the problem into a single one, using either an addition, multiplication or any other combination of arithmetical operations. These functions may be linear or nonlinear.

Table 2.2: Some classical methods.

Method	**Reference**	**Classification**
Global Criterion or Compromise programming	Yu, Zeleny 1973	No-preference
MPB (Multiobjective Proximal Bundle Method)	Mäkelä, Miettinen 1993-1996	No-preference
Weighting Method	Gass and Saaty 1955; Zadeh 1963	A posteriori, a priori
ϵ-constraint	Haimes 1971	A posteriori
Hybrid Method	Corley 1980; Wendell and Lee 1977	A posteriori
Method of Weighted Metrics	Bowman 1976	A posteriori
Achievement Scalarizing Function Approach	Wierzbicki 1980-1986	A posteriori
Hyperplane method	Yano and Sakawa 1989	A posteriori
NISE (Noninferior set estimation)	Cohon 1978	A posteriori
NBI (Normal boundary intersection)	Das and Dennis 1998	A posteriori
Value Function Method	-	A priori
Lexicographic Ordering	-	A priori
Goal Programming	Charnes and Cooper 1955, 1961	A priori
Weighted Goal Programming	Charnes and Cooper 1977	A priori
Lexicographic Goal Programming	-	A priori
Min-Max Goal Programming	Flavell 1976	A priori
ISWT (Interactive Surrogate Worth Trade-Off Method)	Chankong and Haimes 1978	Interactive
GDF (Geoffrion-Dyer-Feinberg)	Geoffrion et al. 1972	Interactive
SPOT (Sequential Proxy Optimization Technique)	Sakawa 1982	Interactive
Tchebycheff Method	Steuer 1986	Interactive
STEM (Step Method)	Benayoun 1971	Interactive
Reference Point Method	Wierzbicki 1980-1982	Interactive
GUESS Method	Buchanan 1997	Interactive
STOM (Satisficing Trade-Off Method)	Nakayama et. Al 1984-1995	Interactive
Light Beam Search	Jaszkiewicz and Slowinski 1994,1995	Interactive
Reference Direction Approach	Korhonen and Laakso 1984-1986	Interactive
RD (Reference Direction Method)	Narula et al. 1994	Interactive
NIMBUS (Nondifferentiable Interactive Multiobjective Bundle-based optimization System)	Miettinen and Mäkelä (1994-1997)	Interactive
Interactive multiple goal programming (IMGP)	Nijkamp and Spronk (1980, 1990)	Interactive
Sequential multiobjective problem solving (SEMOPS)	Monarchi et al. (1973)	Interactive

was made by Wolpert and Macready [60] in 1997. They published the *No Free-Lunch (NFL) theorem*, which is a class of theorems concerning the average behavior of optimization algorithms over a set of optimization problems.

The primary of such theorems states that, if problem domain knowledge is not incorporated into the algorithm domain, no formal assurances of an algorithm's general robust effectiveness exist.

NFL theorems, in addition, imply that incorporating too much problem domain knowledge into a search algorithm reduces its effectiveness on other problems outside and even within a particular class; therefore, robustness is sacrificed in such cases.

In [61], Köppen extends the NFL theorem to the case of multi-objective optimization, which states that, on average, each algorithm has the same performance when applied to all possible sets of problems, provided that no a priori knowledge of the problem is assumed.

There has been some considerable debate about the utility of the NFL theorems, often centered around the question of whether the set of problems that we are likely to tackle with evolutionary algorithms is representative of all problems, or if they form some special subset. However, the NFL theorems have come to be widely accepted, and the following lessons can be drawn from them [62]:

- If an algorithm does particularly well on one class of problems, then it is likely to do more poorly over the remaining problems. This suggests that a careful strategy is required to evaluate algorithms.
- For a given problem, we can circumvent the NFL theorem by incorporating problem-specific knowledge. This of course leads us towards *memetic algorithms*.[6]

2.7 Evolutionary Computation

Evolutionary computation is a subfield of artificial intelligence, which studies computational systems that use ideas and get inspiration from natural evolution and adaptation. It aims at understanding such computational systems and developing more robust and efficient ones for solving complex real-world problems [63].

These computational systems, also known as evolutionary algorithms, are metaheuristics, which are considered, in general, a posteriori methods. They must be used only as a last resort once all other strategies have been exhausted, or when time consumption or results are not satisfactory. Evolutionary algorithms are also recommended in cases where it is not possible to formulate a problem, or when dealing with inaccurate and noisy data.

The aim of this section is to describe what an evolutionary algorithm is, its main components and related terminology. Section 2.7.1 presents the main branches of evolutionary computation and points out similarities and differences among its different paradigms. Additionally, Section 2.7.2 provides an example of a Multi-Objective Evolutionary Algorithm (MOEA), which has been widely studied and applied in different areas of knowledge.

Some of the terms used in evolutionary computation are *individuals*, which are solution candidates; and *population*, which is the set of solution candidates. Each individual represents a possible solution, i.e., a decision vector, which is encoded, using an appropriate representation.

All evolutionary algorithms have three prominent features which distinguish them from other search algorithms. First, they are all population-based. Second, there is communication and information exchange among individuals in a population by means of selection and/or recombination. Third, they are stochastic.

The search mechanism of evolutionary algorithms can be summarized by equation (2.12):

[6] Metaheuristics which incorporate a local search rule with a population-based strategy.

Algorithm 1 Pseudocode for the tournament selection algorithm.

Input: Population P, Tournament size k
Output: The selected individuals to become parents

```
Q ← ∅
while |Q| ≤ |P| do
    Pick k individuals randomly from P, with or without replacement
    Let P_i be the best of these k comparing their fitness values
    Q ← Q ∪ P_i
return Q
```

$$P[t+1] := s_s\left(v\left(s_p(P[t])\right), P[t]\right), \tag{2.12}$$

where $P[t]$ is the population at iteration or generation t, s_p is the operator for parent selection, v is the variation operator, and s_s is the operator for survivor selection. In the following, we explain in detail each component.

The *parent selection* or *mating selection* aims at picking promising solutions based on their quality for variation. This process usually consists of two stages: *fitness assignment* and *sampling*. In the first stage, the individuals in the current population are evaluated in the objective space and then assigned a scalar value, the *fitness*, reflecting their quality. Afterwards, a so-called *mating pool* is created by random sampling from the population according to the fitness values.

An important term, related to this operator, is *selection pressure*, which drives the population towards better solutions. However, when fitness values of individuals are all very close together, there is almost no selection pressure.

For instance, a straightforward sampling method is *tournament selection*, which is shown in Algorithm 1 [62]. Here, two individuals are randomly chosen from the population, and the one with the better fitness value is copied to the mating pool. This procedure is repeated until the mating pool is filled. The complexity of this deterministic version is $\mathcal{O}(|P|)$, since it requires $|P|$ competitions, each of which is performed in $\mathcal{O}(1)$. The selection pressure is controlled by varying the tournament size; the larger the tournament, the higher is the chance that it will contain members of above-average fitness, and the lower that it will consist entirely of low-fitness members.

On the other hand, the *variation operators* take a set of solutions from the mating pool and systematically or randomly modify these solutions in order to generate potentially better solutions. There are usually two variation operators: recombination and mutation. The essence of the *recombination* or *crossover operator* is the inheritance of information (genes) from two or more parents by one or two offspring. To mimic the stochastic nature of evolution, a crossover probability is associated with this operator.

By contrast, the *mutation operator* is applied to one individual and delivers a modified mutant, the child or offspring of it, according to a given mutation rate.

Sometimes, an algorithm needs to escape from local optima and avoid *premature convergence*. In this case, mutation comes into play.

It is worth noting that due to random effects some individuals in the mating pool may not be affected by variation and therefore simply represent a copy of a previously generated solution.

There are numerous recombination and mutation operators which have been proposed for different representations [62]. The most common for multi-objective optimization using real numbers encoding is simulated binary crossover (SBX) and polynomial-based mutation [64, 65, 66].

SBX simulates the working principle of single-point crossover on binary strings [62]. This operator is convenient, because the spread of children solutions around parent solutions can be controlled using a distribution index. A large value of this index allows only near parent solutions to be created, whereas a small value of the index allows distant solutions to be created. Another aspect of this crossover operator is that it allows to have a more focused search when the population is converging.

Similarly, for polynomial-based mutation, the amount of perturbation in a variable can also be controlled by fixing a distribution index. This operator is suitable for SBX.

Finally, the *survivor selection*, also known as *environmental selection* or *replacement*, chooses which individuals among parents and offspring will be allowed in the next generation. This decision is usually based on their fitness values, favoring those with higher quality.

Based on the above concepts, natural evolution is simulated iteratively, until a certain stopping criterion is fulfilled. At the end, the best individuals in the final population represent the outcome of the evolutionary algorithm.

2.7.1 Main Branches

Evolutionary computation is an emerging field which has grown rapidly in recent years. It encompasses several major branches, i.e., evolution strategies, evolutionary programming, genetic algorithms and genetic programming. At a philosophical level, they differ mainly in the way in which they simulate evolution. At the algorithmic level, they differ mainly in their representations of potential solutions and the operators that they use to modify solutions.

In Table 2.3, the main characteristics, similarities and differences of evolutionary algorithms are summarized. Here, it is assumed that the symbol μ represents the population size (which is the same as the number of parents), and λ represents the number of offspring generated from all μ parents. Next, we describe briefly these approaches.

Evolution strategies were first proposed by Rechenberg and Schwefel in 1965 as a numerical optimization technique of parameters (see [67] for a brief history). The original evolution strategy did not use populations. A population was introduced into evolution strategies a few years later [68, 29]. The main contribution of this approach is *self-adaptation*, which includes parameters settings in the evolution pro-

Table 2.3: Main characteristics of evolutionary algorithms.

Feature	**Evolution Strategies**	**Evolutionary Programming**	**Genetic Algorithms**	**Genetic Programming**
Abstract Level	Individual behavior	Species behavior	Organisms	Organisms
Representation	Real-valued	Real-valued	Binary-valued	Tree structures
Self-adaptation	Standard deviation and covariances	None (meta-EP: standard deviation, step sizes, variances; R-meta-EP: covariance matrices)	None	None
Mutation	Main operator (Gaussian perturbation)	Only operator (Gaussian perturbation)	Secondary operator	Secondary, optional (random change in trees)
Recombination	Important, but secondary (discrete or intermediate)	None	Main operator	Main operator Exchange of subtrees
Parent Selection	Probabilistic (uniform distribution)	Deterministic (each parent creates one offspring via mutation)	Probabilistic, biased by fitness (preservative)	Probabilistic, biased by fitness (preservative)
Survivor Selection	- Extinctive: (1+1), $(1+\lambda)$, $(\mu+1)$ - Deterministic, biased by fitness: (μ,λ), $(\mu+\lambda)$	Probabilistic, biased by fitness $(\mu+\mu)$	- Generational replacement - Deterministic, biased by fitness: $(\mu+\lambda)$	- Generational replacement (simple GA) - Probabilistic, biased by fitness (steady-state GP)

cess of individuals. There are five major selection schemes in evolution strategies:

- $(1+1)$: one parent generates one child, and the best of them is selected to form the next generation.
- $(1+\lambda)$: one parent generates λ offspring. The best individual from the parent and the λ offspring is selected to form the next generation.
- $(\mu+1)$: μ parents generate one child, which can substitute the worst parent.
- $(\lambda+\mu)$: λ offspring are generated from μ parents. The μ fittest individuals from $\lambda+\mu$ candidates are selected to form the next generation.
- (λ,μ): the μ fittest individuals from λ $(\lambda \geq \mu)$ offspring are selected to form the next generation.

The first three schemes use an *extinctive selection* mechanism, since some individuals are excluded from being selected for reproduction, while the last two schemes use a *deterministic selection* approach, since they always choose the best solutions from the population.

Evolutionary programming was first proposed by Fogel et al. in the mid 1960s as a way to achieve artificial intelligence [69, 70]. This approach is similar to evolution strategies, however it does not use any recombination. Moreover, μ parents generate μ offspring, from which μ individuals are selected using a pairwise tournament competition.

Genetic algorithms were initially conceived by Holland [71] and his students [72] in 1975 although some of the ideas appeared as early as 1957 in the context of simulating genetic systems [73]. It is probably the most well-known branch of evolutionary computation, which emphasizes genetic encoding of potential solutions into chromosomes. In an attempt to prevent the loss of good solutions during the optimization process due to random effects, *elitism* can be incorporated. In essence, a trace is kept of the current fittest members, and such individuals are always kept in the population. Additionally, the genetic algorithm uses *preservative selection*, since each individual has a non-zero probability of being selected as a parent.

A special sub-branch of genetic algorithms is genetic programming. These algorithms evolve tree-structured chromosomes. They were first used by Koza in 1989 [74, 75].

2.7.2 A sample MOEA: NSGA-II

A very popular evolutionary algorithm, proposed by Deb et al. [13] in 2000, is the Nondominating Sorting Genetic Algorithm II, which is an improved version of the Nondominated Sorting Genetic Algorithm (NSGA) [76, 77], and it is inspired by the ranking procedure originally proposed by Goldberg [78] in 1989. Here, the population is partitioned into *layers* or *fronts* using a nondomination criterion.

This ranking procedure is shown in Algorithm 2. In lines 1 to 8, for each individual p, the set of solutions that p dominates (S_p), and the number of solutions which dominate p (n_p) are calculated. This requires $\mathcal{O}(mN^2)$, where m is the number of objectives, and $N = |P|$ represents the population size.

In lines 9 to 11, all the individuals with domination count as zero belong to the first nondominated front. In lines 13 to 20, for each solution p with $n_p = 0$, each member q of its set S_p is visited and its domination count is reduced by one. In doing so, if for any member q, the domination count becomes zero, then it is placed in a separate list Q. At the end of the iterative cycle (lines 21 and 22), the members of this list belong to the second nondominated front. Now, the above procedure is continued with each member of Q and the third front is identified. This process continues until all fronts are identified.

For each solution p in line 15, the domination count n_p can be at most $N - 1$. Thus, each solution p will be visited at most $N - 1$ times before its domination count becomes zero. At this point, the solution is assigned a nondomination level and will never be visited again. Since there are at most $N - 1$ such solutions, the total complexity is $\mathcal{O}(N^2)$. Thus, the overall complexity of this procedure is $\mathcal{O}(mN^2)$, and the storage requirement is $\mathcal{O}(N^2)$.

Diversity is maintained by the *crowding distance*, which estimates the density of solutions surrounding a particular solution p in the population. This quantity, *p.dist*,

Algorithm 2 Pseudocode of *nondominated-sort*

Input: Population P

Output: Partition in fronts of the population

```
for all p ∈ P do
    S_p ← ∅
    n_p ← 0
    for all q ∈ P do
        if (p ≺ q) then
            S_p ← S_p ∪ {q}
        else if (q ≺ p) then
            n_p ← n_p + 1
    if n_p = 0 then
        p.rank ← 1
        F_1 ← F_1 ∪ {p}
i ← 1
while F_i ≠ ∅ do
    Q ← ∅
    for all p ∈ F_i do
        for all q ∈ S_p do
            n_q ← n_q − 1
            if n_q = 0 then
                q.rank ← i + 1
                Q ← Q ∪ {q}
    i ← i + 1
    F_i = Q
return F
```

corresponds to the average distance of two points on either side of this point along each of the objectives.

In Algorithm 3, we present the pseudocode to calculate the crowding distance of a set of individuals. First, the distance is initialized for all individuals. Then, in lines 3 to 4, the set of individuals is sorted with respect to each objective function value. In line 5, the extreme solutions are assigned an infinite distance value. In lines 6 and 7, the distance values are calculated for all other intermediate solutions. This calculation is continued with other objective functions. The overall crowding-distance value is calculated as the sum of individual distance values corresponding to each normalized objective function.

Here, $P_j.f_i$ refers to the i-th objective function value of the j-th individual in the set P. The complexity of this procedure is governed by the sorting algorithm. Since m independent orderings of at most N solutions (when all population members are in one front P) are involved, this algorithm has $\mathcal{O}(mN \log N)$ computational complexity.

The main loop of NSGA-II is shown in Algorithm 4. First, the population is initialized at random and evaluated in each objective function. Then, the upper

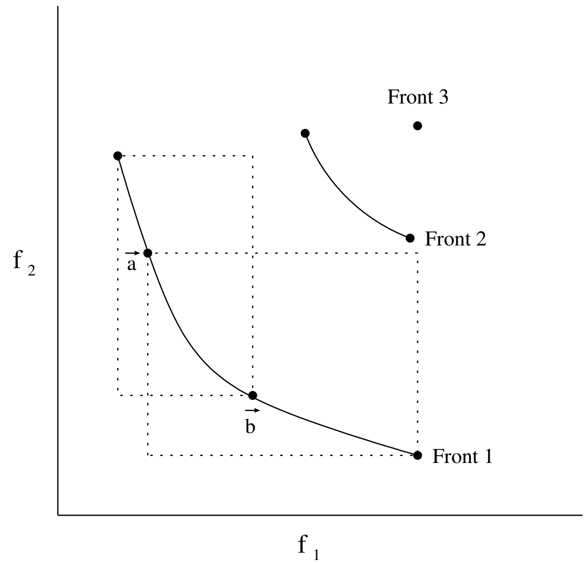


Figure 2.10: Schematic of fronts and crowding distance of individuals in NSGA-II. The fronts are represented by the continuous lines. The dashed regions depict the crowding distance of solutions $\vec{a}$ and $\vec{b}$. Here, solution $\vec{b}$ is preferred over $\vec{a}$. The boundary solutions of the first front have infinite crowding distance.

Algorithm 3 Pseudocode of the *crowding-distance*

Input: Set of individuals P, vector of maximum and minimum values of each objective function $\vec{f}^{min}$, $\vec{f}^{max}$

Output: Assignment of the crowding distance

$l \leftarrow |P|$
$(\forall p \in P)\ p.dist \leftarrow 0$
for all $i \in \{1, \ldots, m\}$ **do**
    Sort individuals P according to the i-th objective value in ascending order
    $P_1.dist \leftarrow P_l.dist \leftarrow \infty$
    for $j = 2$ to $l - 1$ **do**
        $P_j.dist \leftarrow P_j.dist + (P_{j+1}.f_i - P_{j-1}.f_i)\,/\,(f_i^{max} - f_i^{min})$

and lower bounds of all objective functions are obtained in line 4. Each solution is assigned a fitness (or rank) equal to its nondomination level in line 5. At each generation, tournament selection with $k = 2$ (see Algorithm 1) is executed for the current population in line 7. In line 8, recombination and mutation operators are used to create an offspring population of size N. This offspring is evaluated in line 9, and the minimum and maximum values of each objective function are updated in line 10. Since elitism is introduced, both populations of parents and offspring are sorted with respect to Algorithm 2 in line 11. Then, in lines 12 to 17, the solutions of the lower fronts are considered in the next generation. If the last front is larger than the population size, then its solutions are sorted according to their rank and

Algorithm 4 Pseudocode of NSGA-II

Input: MOP, termination condition, population size N
Output: Approximation to the Pareto optimal set
1: $t \leftarrow 0$
2: Initialize population P_t
3: Evaluate population P_t
4: Obtain $\vec{f}^{min}$ and $\vec{f}^{max}$
5: *nondominated-sort*(P_t)
6: **while** termination condition is not fulfilled **do**
7: Perform binary tournament selection
8: Generate offspring P'_t using variation operators
9: Evaluate population P'_t
10: Update $\vec{f}^{min}$ and $\vec{f}^{max}$
11: $F \leftarrow$*nondominated-sort*$(P_t \cup P'_t)$
12: $P_{t+1} \leftarrow \emptyset$
13: $i \leftarrow 1$
14: **while** $|P_{t+1}| + |F_i| \leq N$ **do**
15: *crowding-distance*$(F_i, \vec{f}^{min}, \vec{f}^{max})$
16: $P_{t+1} \leftarrow P_{t+1} \cup F_i$
17: $i \leftarrow i + 1$
18: Sort individuals of the front F_i
19: $P_{t+1} \leftarrow P_{t+1} \cup F_i[1 : (N - |P_{t+1}|)]$
20: $t \leftarrow t + 1$
21: **return** P_t

crowding distance (line 18). The best solutions fill the remaining population (line 19). The overall complexity of the algorithm is $\mathcal{O}(mN^2)$, which is governed by the nondominated sorting part of the algorithm.

It is worth noting that when we compare two solutions in tournament selection or sorting of the population, we prefer the solution with the lower (better) rank. Otherwise, if both solutions belong to the same front, then we prefer the solution that is located in a less crowded region.

In Figure 2.10, we illustrate the front building process, as well as the crowding distance for a hypothetical population. As we will see in the next section, NSGA-II is not effective in solving optimization problems with more than three objectives.

2.8 Many-Objective Optimization

Evolutionary algorithms based on Pareto optimality have proven to be very useful since the 1990s, but recent studies have shown that they do not perform properly when dealing with problems having four or more objectives [1, 2]. Farina and Amato [79] in 2002 were the first in observing this behavior and introduced the terminology *many-objective*, which was extended to *many-objective optimization* by Purshouse and Fleming [80] a year later. This area is currently considered a hot research topic.

The goal of this section is to present a general overview of many-objective optimization, including: main issues, alternative solutions, and examples of real-world problems.

According to Ishibuchi et al. [2], the difficulties that we may encounter in MOEAs when applying Pareto dominance to many-objective problems are the following:

1. **Deterioration of the search ability**. When the number of objectives increases, almost all solutions in each population become nondominated. This severely weakens the Pareto dominance-based selection pressure toward the Pareto optimal front. Therefore, the convergence property of MOEAs is severely deteriorated.

2. **Exponential increase in the number of solutions required for approximating the entire Pareto optimal front**. The goal of MOEAs is to find a set of nondominated solutions that properly approximates the entire Pareto optimal front. Since the Pareto optimal front is a hyper-surface in objective space, the number of solutions required for its approximation exponentially increases with the dimensionality of objective space. Therefore, we may need thousands of nondominated solutions to approximate the entire Pareto optimal front of a many-objective problem.

3. **Difficulty of the visualization of solutions**. It is usually assumed that the choice of a final solution from a set of obtained nondominated solutions is done by a decision maker based on his/her preference. The increase in the number of objectives makes the visualization of obtained nondominated solutions very difficult. This means that the choice of a final solution becomes very difficult in many-objective optimization.

The deterioration of the search ability is produced mainly by the phenomenon of *dominance resistance*,[7] which is the difficulty in producing new candidate solutions that will dominate current solutions. Deb et al. [83] and Hanne [81] also identified that the level of dominance resistance could increase with the dimension of objective space. The deterioration of convergence has been pointed out in a number of studies [80, 84, 85], where the bad performance of NSGA-II has been demonstrated.

Recent advances in the field of evolutionary many-objective optimization led to the conclusion that the addition of an objective makes the problem indeed harder, but, it can be argued that the difference is not significant [86].

According to Bentley et al. [87] the number of nondominated m-dimensional vectors on a set of size N is $O(\ln^{m-1} N)$. As a consequence, in many-objective problems, the selection of solutions is carried out almost at random or guided by diversity criteria. In fact, Knowles and Corne [1] demonstrated that, for MOPs with more than

[7] This term was first identified by Hanne [81] and Ikeda et al. [82] and, in the context of many-objective optimization by Deb et al. [83].

ten objectives, a purely random search may perform favorably when compared with a MOEA.

A straightforward idea for the scalability improvement of MOEAs to many-objective problems is to increase the selection pressure toward the Pareto optimal front. One approach based on this idea is to modify Pareto dominance in order to decrease the number of nondominated solutions in each population [88]. Another approach is to assign different ranks to nondominated solutions [89, 90, 91, 92].

Another idea for the scalability improvement is the use of different fitness evaluation mechanisms (instead of Pareto dominance). One approach based on this idea is to use a number of different scalarizing functions for fitness evaluation [93, 85, 94, 95, 14]. Another approach is the use of indicator-based evolutionary algorithms where indicator functions are used to evaluate each solution [96, 9, 15].

On the other hand, for the second difficulty, the number of points necessary to represent a Pareto optimal front is bounded by $O(mr^{m-1})$, where r is the resolution and m the number of objectives [97]. This issue has often been tackled by incorporating preference information in MOEAs [41, 98, 99, 100]. As we have seen in Section 2.2, for multi-objective optimization the desirable aspects of nondominated sets are convergence and diversity. Moreover, for many-objective optimization, the property of *pertinency* [101] or *preferability* [102] has special prominence. This feature consists of producing solutions that reside in the decision maker's region(s) of interest. In practice, and especially as the number of objectives increases, the decision maker is interested only in a sub-region of objective-space. Thus, focusing on pertinent areas of the search space helps to improve optimizer efficiency and reduces unnecessary information that the decision maker would otherwise have to consider. At the end of the optimization process, the decision maker can then decide on a single solution to be implemented based on preferences and application dependent high-level information.

A direct approach for handling the difficulty of visualization is to decrease the number of objectives [103, 104, 105, 106]. Of course, dimensionality reduction can remedy not only the third difficulty but also the other difficulties. Visualization techniques of nondominated solutions with many objectives have been proposed in the literature [107, 108] where objective vectors are mapped into a low-dimensional space for their visualization. A number of visualization techniques of high-dimensional objective vectors have also been proposed in the field of multiple criteria decision making [32].

One practical method that will be used in this document is the *parallel coordinates* or *value paths* [109, 110], which represent sets of objective vectors. To show a set of points in an m-dimensional objective space, a backdrop is drawn consisting of m parallel lines, typically vertical and equally spaced. An objective vector is represented as a poly-line with vertices on the parallel axes; the position of the vertex on the ith axis corresponds to the ith value of the objective function. If the ranges are known, they give additional information about the possibilities and limitations of the objective functions. It is worth noticing, that each objective function can have a scale of its own. Parallel coordinates are a recommended visualization method because they are easy to interpret. For example, it is easy to distinguish non-Pareto optimal alternatives if they are included. Further, even a large number of objective functions causes no problem.

According to López [111], the number of function evaluations is another important difficulty when dealing with many-objective optimization problems, since some real-world problems have a small budget of function evaluations, due to time constraint reasons. Some memetic algorithms have been proposed to tackle this issue using less than 1000 function evaluations [112, 113, 114, 115]. Other challenges are related to the design of both data structures to efficiently manage the population, and density estimators to achieve an even distribution of the solutions along the Pareto optimal front. Unfortunately, even if we could efficiently obtain an accurate approximation of the Pareto optimal front, the selection of one solution among such a huge number of solutions would be a very difficult task for the decision maker.

In Tables 2.4 and 2.5 [30], we provide several examples of real-world many-objective optimization problems. These diverse applications require from 4 up to 500 objectives, being engineering the area in which they seem to appear more frequently.

2.9 Summary

This chapter introduced the basic concepts of multi-objective optimization, which is widely used in several disciplines. We started by defining a MOP, the concept of optimality, and the relations of normal, weak and strong dominance. Moreover, we denote that the outcome of an optimizer is the Pareto set approximation, which should meet the desirable aspects of convergence, distribution and spread.

We defined three important points in objective space: ideal, utopian and nadir vectors, which are used as reference solutions in multi-objective algorithms. Then, we mentioned that the decision maker is the only person who is supposed to express preference relations among the different solutions produced by an optimizer.

We discussed the features that are present in MOPs, in both search space and fitness space, and the methods that have been proposed to solve them, which are classified as enumerative, deterministic and stochastic; or categorized according to the participation of the decision maker: non-preference, a posteriori, a priori, and interactive. We introduced the No Free-Lunch theorem, which states that if problem domain knowledge is not incorporated into the algorithm domain, no formal assurances of an algorithm's general robust effectiveness exist.

This chapter also gave a quick overview of evolutionary computation. We discuss its main components, and related terminology, such as, individual, population, parent selection, fitness, selection pressure, variation operator, crossover, mutation, offspring, premature converged and reduction.

Additionally, we presented a brief history of major types of evolutionary algorithms, i.e., evolution strategies, evolutionary programming, genetic algorithms, and genetic programming.

Finally, we gave a general overview of many-objective optimization, its main issues,

Table 2.4: Real-world applications for many-objective optimization.

Field	Specific Application	Reference(s)	#	Objectives
Environmental Engineering	Groundwater pollution remediation	Garrett et al. (1999) [116]	4	Aquifer pumping flow, oxygen injection rate, toluene pulsing rate, and separation.
Environmental Engineering	Location of siting retail and service facilities	Guimarães et al. (1993-94) [117, 118]	5	Height, geology, aspect, land use, and distance from two urban centers.
Electrical and Electronics Engineering	Synthesis of CMOS operational amplifiers	Zebulum et al. [119]	7	Gain, linearity, power consumption, area, phase margin, slew-rate, and GBW.
Telecom. and Network Optimization	Improve wire-antenna geometries	Van Veldhuizen et al. [120]	4	Radiated power gain, azimuthal symmetry of radiated power, input resistance, and input reactance.
Robotics and Control Engineering	Controller design	Schroder et al. [121]	9	Steady state error, compliance, maximum current, noise susceptibility, complexity, etc.
Robotics and Control Engineering	Controller design	Donha et al. [122]	4	Overshoot, controller roll-off frequency, rise and settling times.
Robotics and Control Engineering	Robust controller design	Herreros López and co-workers [123, 124, 125]	4	Related to mixed H/H_∞ controller design.
Robotics and Control Engineering	Design of control systems for a gas turbine engineering	Chipperfield and Fleming (1995-96) [126, 127]	9	Related to the foreaft differential and total engine thrust.
Robotics and Control Engineering	Design of control systems	Dakev et al. and Chipperfield et al. (1995-96) [128, 129]	7	Performance parameters of the electromagnetic suspension system for a maglev vehicle.
Robotics and Control Engineering	Design of control systems	Tan and Li (1997) [130]	9	Stability, closed-loop sensitivity, disturbance rejection, plant uncertainty, actuator saturation, rise and settling times, overshoots, and steady state error.
Robotics and Control Engineering	Robotic manipulator problem	Jakob et al. (1992) [131]	5	Failure checks (overstep and collisions), accuracy in reaching the target position, smooth path, travel time, energy consumption.
Robotics and Control Engineering	Robotic manipulator problem (trajectory)	Ortmann and Weber (2001) [132]	6	Related to the joints of a robot arm.
Robotics and Control Engineering	Robotic manipulator problem (counter-weight balancing)	Coello et al. (1995;1998) [133]	4	Torques and joint forces of the robot arm.
Structural and Mechanical Engineering	Micromechanical modeling parameters	Reardon (1998) [134]	17	Related to experimental data points.
Transport Engineering	Road systems (alternative motorway routes)	Guimarães et al. (1995-97) [135, 136, 137]	5	Height, geology, aspect, land use, and distance from urban centers.
Transport Engineering	Road systems (train)	Laumanns et al. (2001) [138]	10	Weight, gear box, engine and driving strategy, fuel consumption, driving performance, and convenience, etc.
Transport Engineering	Road systems	Qiu (1997) [139]	17	Maximize the investment effectiveness subject to the current budget constraints.
Aeronautical Engineering	Helicopter design	Flynn and Sherman (1995) [140]	4	Buckling of panel and bay, panel weight, number of frames and stiffeners.
Aeronautical Engineering	Aerodynamic optimization (subsonic wing design)	Anderson (1995) [141]	4	Lift/drag ratio, lift/weight ratio, meet design lift goals, and maintain structural integrity.
Chemistry	Intensities of emission lines of trace elements	Wienke et al. (1992) [142]	7	Combination of relative emission intensities.

Table 2.5: Real-world applications for many-objective optimization (continuation).

Field	Specific App.	Reference(s)	#	Objectives
Chemistry	Polymer extrusion optimization	Gaspar Cunha et al. (1997) [143]	4	Melt temperature, length of screw required for melting, power consumption and mass output.
Medicine	Treatment planning in radiation therapy	Yu (1997) [144]	4	Maximum dose in the target volume, average dose above the mean in the anterior critical structure, average dose above the mean in the posterior critical structure and average dose above the mean in the normal tissue shell.
Medicine	Left ventricle 3D reconstruction	Aguilar and Miranda (1999) [145]	6	Slice fidelity, internal energy of the reconstructed slice, and energy of similarity between the current slice configuration and the adjacent slice previously reconstructed.
Ecology	Assessment of ecological models	Reynolds and Ford (1999) [146, 147]	10	Mortality, stand height frequency distribution, median live tree height, number of live whorls, crown angle and length ratio, suppressed tree growth rate, variability in suppressed tree height increment rates, dominant tree slope and variability in dominant tree height increment rates.
Computer Science and Engineering	Coordination of agents	Cardon, Galinho et al. (1998-2000) [148, 149, 150]	up to 500	There are as many objectives as jobs in a Gantt diagram for scheduling problems.
Computer Science and Engineering	Games	Chow (1998) [151]	4	Number of red, white, and blue chromatic rectangles from a 10×10 chessboard, and a distribution factor for the three colors.
Computer Science and Engineering	Image processing	Bhanu and Lee (1994) [152]	5	Edge-border coincidence, boundary consistency, pixel classification, object overlap and contrast.
Computer Science and Engineering	Computer-generated animation	Shibuya et al. (1999) [153]	4	Change of the joint torques, joint torques, acceleration of the handled object, and completion time of the motion.
Computer Science and Engineering	Graph layout generation	Barbosa and Barreto (2001) [154]	5	Uniform spatial distribution of vertices, number of edge-crossings, uniform edge length, exhibition of symmetric features, and avoid having vertices too close to edges.
Design and Manufacture	Machine design (four-cylinder gasoline engine)	Fujita et al. (1998) [155]	4	Maximize miles/fuel covered, acceleration performance, starting response, and follow-up response.
Design and Manufacture	VLSI (building block placement problem)	Esbensen and Kuh (1996a) [156]	4	Maximum path delay, layout area, routing congestion, and aspect ratio deviation with respect to a certain target.
Design and Manufacture	Process planning	Groppetti and Muscia (1997) [157]	6	Assembly cost and cycle-time, product reliability, maintenance costs, product flexibility, and redesign and/or modification flexibility.
Design and Manufacture	Machine design (gas turbine engine)	Fonseca and Fleming (1995;1998) [158, 159]	7	Pole magnitude, gain margin, phase margin, rise and settling times, maximum value reached by the output, output error.
Design and Manufacture	Process planning	Chen and Ho (2001) [160, 161]	4	Total flow time, deviations of machine workload, the greatest machine workload, and the tool costs.
Design and Manufacture	Machine design	Coello and Christiansen (1996;1999) [162, 163]	4	Surface roughness, surface integrity, tool life, and metal removal rate.
Scheduling	Time-tabling	Paechter et al. (1998) [164]	12	Room changes, time restrictions, etc.
Finance	Economic models	Mardle et al. (1998;2000) [165, 166]	4	Maximize profit, maintain quota shares among countries, maintain employment in the industry and minimize discards.

alternative solutions, and examples of real-world problems. An additional desirable aspect in many-objective optimization is pertinency, which consists of producing solutions that reside in the decision maker's region(s) of interest.

Chapter 3

Performance Indicators Used as a Selection Mechanism

Many MOEAs implement a combination of Pareto dominance on sets and a diversity measure based on Euclidean distance in the objective space. While these methods have been successfully employed in various bi-objective optimization scenarios, they appear to have difficulties when the number of objectives increases [96]. As a consequence, researchers have tried to develop alternative concepts, and a recent trend is to use performance indicators (also denoted as quality measures or quality indicators) for search.[1] So far, they have mainly been used for performance assessment. Of particular interest in this context is the $R2$ indicator [167] as it is almost compatible with Pareto dominance and highly correlated with the hypervolume [168, 169], but has a lower computational overhead than such indicator [12].

In this chapter, we overview three performance indicators that can be successfully incorporated as search engine into a MOEA. Moreover, we describe some of its derived algorithms, including their pseudocode and complexity analysis. Firstly, in Section 3.1, we describe the main properties of quality measures. In Section 3.2 we present the hypervolume. In Section 3.3, we introduce a new performance indicator, Delta p. Followed by the $R2$ indicator in Section 3.4. Finally, in Section 3.5 we briefly discuss some other techniques that can be applied.

3.1 Properties

Performance indicators serve different goals: they may be used for comparing the outcomes of multi-objective optimizers, but also during the optimization process as guidance for the search or as stopping criterion. In the following, we provide some important definitions related to performance indicators.

[1] Performance indicators are often referred to in the literature as metrics. However, metric is a well-defined terminology in mathematics and many performance indicators in multi-objective optimization do not necessarily satisfy the conditions for a metric.

Definition 3.1.1. A *(unary) performance indicator* is a function $I : \mathcal{Z} \subset \mathbb{R}^m \rightarrow \mathbb{R}$ that assigns each approximation set a real number.

Definition 3.1.2. An indicator I is said to be *strictly monotonic* if and only if whenever a Pareto set approximation entirely dominates another one, then the indicator value of the dominant set will also be better. Formally, this can be expressed as:

$$\forall A, B \in \mathcal{Z} : A \prec B \Rightarrow I(A) > I(B),$$

where $\prec$ stands for the underlying Pareto dominance of sets.

Definition 3.1.3. An indicator I is said to be *weakly monotonic* if and only if for any Pareto set approximation that is compared to another Pareto set approximation holds: at least as good in terms of the dominance relation implies at least as good in terms of the indicator values. Formally, this can be expressed as:

$$\forall A, B \in \mathcal{Z} : A \preceq B \Rightarrow I(A) \geq I(B),$$

where $\preceq$ stands for the underlying weak-Pareto dominance of sets.

3.2 Hypervolume Indicator

The hypervolume indicator or S metric is one of the most popular set quality measures, since it is the only unary indicator known to be strictly monotonic with respect to Pareto dominance and thereby guaranteeing that the Pareto optimal front achieves the maximum hypervolume possible, while any worse set will be assigned a worse indicator value [28, 170, 6]. In fact, its maximization also leads to sets of solutions whose spread along the Pareto optimal front is maximized (although this does not necessarily mean that such solutions will be uniformly distributed along the Pareto optimal front).

The hypervolume measures the size of the portion of objective space that is dominated by an approximation set [4] (see Figure 3.1). Based on a reference point $\vec{z}$, the hypervolume of an approximation set A is defined as:

$$I_{HV}(A : \vec{z}) = \mathcal{L}\left(\bigcup_{\vec{a} \in A} \left\{\vec{a'} \mid \vec{a} \prec \vec{a'} \prec \vec{z}\right\}\right), \tag{3.1}$$

with $\mathcal{L}(.)$ denoting the Lebesgue measure of a set [171]. The greater the indicator value, the better the approximation set.

The hypervolume contribution of an individual solution reflects the influence of a single point on the quality of the approximation set. Given a solution $\vec{a} \in A$, it is defined as:

$$C_{HV}(\vec{a}, A : \vec{z}) = I_{HV}(A : \vec{z}) - I_{HV}(A \setminus \{\vec{a}\} : \vec{z}). \tag{3.2}$$

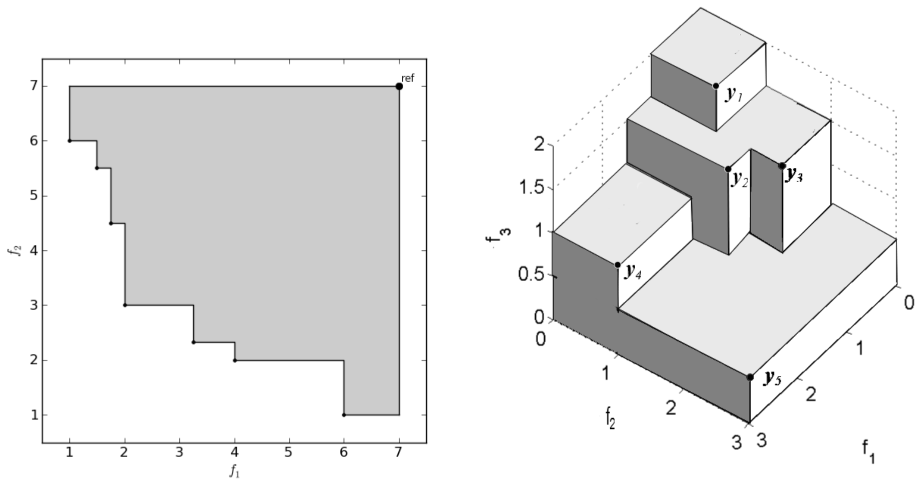


Figure 3.1: Examples of the dominated hypervolume in two and three dimensions. The left figure shows a minimization case whereas the right one shows maximization.

Table 3.1: Hypervolume algorithms. Here, N is the number of solutions considered, m the number of objectives and S represents the number of samples.

Algorithm	**Reference**	**Computational Complexity**
Slicing Objectives (recursive)	Zitzler, 1998 [172] Knowles, 2002 [173] While et al., 2006 [174]	$\mathcal{O}(N^{m-1})$
Dimension-sweep	Fonseca et al., 2006 [175]	$\mathcal{O}(N^{m-2} \log N)$
Related to the Klee measure problem	Beume and Rudolph, 2006 [176] Beume, 2009 [177]	$\mathcal{O}(N \log N + N^{m/2})$ $\mathcal{O}(N^{m/2} \log N)$
Exclusive hypervolume	While et al., 2012 [178]	$\mathcal{O}(N^m)$
Approximated by means of Monte Carlo Simulation	Everson et al., 2002 [179] Bader et al., 2010 [180, 10] Bringmann and Friedrich, 2008 [181, 11]	$(\mathcal{O}(SN^2m))$

The nice mathematical properties of the hypervolume indicator has triggered an important amount of research, including work that focuses on computing it in a more efficient way (see Table 3.1).[2] Even though, it has been shown, that there is no polynomial algorithm available to compute the hypervolume, it is possible to approximate the hypervolume contribution, significantly reducing its computational cost. However, few studies of the performance of such approaches with respect to those using exact hypervolume calculations are currently available.

Several hypervolume-based MOEAs have been proposed in the literature [7, 21, 106, 10]. In the following subsections, we describe two relevant algorithms: IBEA

[2] The tightest known lower bound for 2 or 3 objectives is of order $\Omega(NlogN)$.

(Subsection 3.2.1) and SMS-EMOA (Subsection 3.2.2).

3.2.1 IBEA

The Indicator Based Evolutionary Algorithm, proposed by Zitzler and Künzli[182], is a general framework to incorporate performance indicators in a MOEA. The fitness of a solution is based on the sum of the indicator values resulting from pairwise comparisons to all other solutions.

Several variants of IBEAs have been proposed in the literature [44-46]. In those variants of IBEAs, hypervolume was almost always used as an indicator. Wagner et al. [18] reported good results by IBEAs for many-objective problems. Since IBEAs do not use Pareto dominance, their search ability is not severely deteriorated by the increase in the number of objectives. One difficulty in the application of IBEAs to many-objective problem is a large computation cost for hypervolume calculation. Ishibuchi et al. [19] proposed an iterative version of IBEAs to decrease the computation cost by searching for only a small number of representative solutions. Objective reduction in IBEAs was examined in Brockhoff and Zitzler [28] for the same purpose.

3.2.2 SMS-EMOA

The S Metric Selection-Evolutionary Multi-objective Optimization Algorithm, proposed by Beume, et al. [3] in 2006, combines ideas borrowed from other MOEAs, like the NSGA-II and archiving strategies presented by Knowles et al [183, 184]. It is an algorithm founded on two pillars: (1) nondominated sorting is used as a ranking criterion (see Algorithm 2) and (2) the hypervolume is applied as selection criterion to discard that individual, which contributes the least hypervolume to the worst-ranked front.

The MOEA is described in Algorithm 5. Starting with an initial population of N individuals, a new individual is generated by means of randomized variation operators. The new individual will become a member of the next population, if replacing another individual leads to a higher quality of the population with respect to the hypervolume indicator. Afterwards, one individual is discarded from the worst ranked front. Whenever this front comprises $|R_h| > 1$ individuals, the individual $x \in R_h$ is eliminated that minimizes the hypervolume contribution.

SMS-EMOA guarantees that the covered hypervolume of a population cannot decrease in future generations. Moreover, the produced solutions are well-distributed on the Pareto optimal front. Boundaries of the Pareto optimal front and regions around knee points (see Definition 2.5.14) are favored. A distinguishing feature of the SMS-EMOA is that it is well-suited for approximating Pareto sets by a small number of individuals. The computational effort required for computing hypervolume is very high for more than three objectives. Therefore, the SMS-EMOA is hardly applicable to higher dimensional problems that normally desire a large number of function evaluations.

Algorithm 5 Pseudocode of the SMS-EMOA

Input: MOP, stopping criteria, ideal objective vector $\vec{z}$
Output: Pareto set approximation

Initialize population P
Evaluate population P
while stopping criteria is not fulfilled **do**
    Generate offspring p from P using variation operators
    Evaluate individual p
    $P \leftarrow P \cup \{p\}$
    $\{R_1, \ldots, R_h\} \leftarrow$ nondominated-sort(P)
    $\forall x \in R_h : r(x) \leftarrow C_{HV}(\vec{x}, R_h : \vec{z})$ (see eq. (3.2))
    $x^* \leftarrow \arg\min_{x \in R_h} r(x)$
    $P \leftarrow P \setminus \{x^*\}$
return P

3.3 Delta p Indicator

More recently, a new performance indicator called Δ_p was proposed in [22]. This indicator can be seen as an "averaged Hausdorff distance" between the outcome set and the Pareto optimal front. It is composed of slight modifications of two well-known performance indicators: generational distance (GD, see [23]) and inverted generational distance (IGD, see [24]).

Definition 3.3.1. Given an approximation set A and a discretized Pareto optimal front PF of a MOP, the (slightly modified) GD indicator is defined as:

$$I_{GD_p} = \left(\frac{1}{|A|} \sum_{i=1}^{|A|} d_i^p \right)^{1/p}, \tag{3.3}$$

where d_i is the Euclidean distance from $a_i \in A$ to its nearest member of PF.

Definition 3.3.2. Given an approximation set A and a discretized Pareto optimal front PF of a MOP, the (slightly modified) IGD indicator is defined as:

$$I_{IGD_p} = \left(\frac{1}{|PF|} \sum_{i=1}^{|PF|} d_i^p \right)^{1/p}, \tag{3.4}$$

where d_i is the Euclidean distance from $pf_i \in PF$ to its nearest member of A.

Both I_{GD_p} and I_{IGD_p} have (weak) metric properties:

- I_{GD_p} and I_{IGD_p} are non-negative
- I_{GD_p} and I_{IGD_p} are non-symmetric

- I_{GD_p} and I_{IGD_p} do not satisfy the (relaxed) triangle inequality.

Definition 3.3.3. Given an approximation set A and a discretized Pareto optimal front PF of a MOP, the Δ_p indicator is defined as:

$$I_{\Delta_p} = max(I_{GD_p}, I_{IGD_p}) \tag{3.5}$$

The Δ_p indicator has better metric properties than either the GD or IGD indicators:

- It is positive and symmetric: I_{Δ_p} is a semi-metric.
- If the magnitudes of the sets are bounded, the relaxed triangle inequality is satisfied and I_{Δ_p} is a pseudometric.
- If $p = \infty$ then I_{Δ_p} is a metric (the Hausdorff distance).

Δ_p simultaneously evaluates proximity to the Pareto optimal front and spread of solutions along it. Although Δ_p is not Pareto compliant, its computation has a much lower computational cost than that of the hypervolume, and it can also handle outliers, which makes it attractive for assessing performance of MOEAs.

It is worth noting, however, that for incorporating Δ_p into the selection mechanism of a MOEA, it is necessary to have an approximation of the true Pareto optimal front at all times. This has motivated the development of techniques that can produce such an approximation in an efficient and effective way. In Sections 3.3.1 and 3.3.1 we review two pioneer approaches.

3.3.1 Delta p-EMOA

In [25], the authors linearize the nondominated (piecewise linear) front of the current population, and include this mechanism in the Δ_p-EMOA, which is used for solving bi-objective optimization problems. This algorithm is inspired by SMS-EMOA, and is assisted by a secondary population. Δ_p-EMOA performs better than NSGA-II [13], while consuming a lower number of function evaluations. An extension of this approach to three-objective problems is reported in [26]. In this case, the algorithm requires some previous mathematical steps which include reducing the dimensionality of the nondominated solutions and calculating their convex hull. This version of Δ_p-EMOA achieves a better distribution of solutions than MOEA/D [14], SMS-EMOA and NSGA-II. However, this MOEA requires additional parameters and consumes a high computational time when dealing with many-objective optimization problems.

3.3.2 Delta p-DDE

Another possible approach to incorporate Δ_p into a MOEA is to use an echelon form of the nondominated individuals for the Pareto optimal front. This is the mechanism adopted in Δ_p-DDE [9], in which Δ_p is used as the selection mechanism of a differential evolution algorithm. Δ_p-DDE was able to outperform NSGA-II and provided competitive results with respect to SMS-EMOA, but at a considerably lower computational cost for many-objective optimization problems. The main limitation of this approach is that it produces a poor spread of solutions in high-dimensional search spaces. Also, it has some difficulties for dealing with discontinuous Pareto optimal fronts.

In Algorithm 6, we present the pseudocode of this MOEA. Here, the individual contributions are defined as follows:

Definition 3.3.4. The contribution of an individual a to the GD indicator (C_{GD}) can be defined in a straightforward manner as:

$$C_{GD}(a, A, PF) = d_i, \tag{3.6}$$

where d_i is the Euclidean distance from a_i to its nearest member of PF.

Definition 3.3.5. Let Q be the set of all elements of the Pareto optimal front PF for which $a_i \in A$ is the closest element in A. The contribution of a_i to C_{IGD} can be defined as:

$$C_{IGD}(a_i, A, PF) = \begin{cases} \sqrt[p]{\sum_{q \in Q} dist(q, a_i)^p} & \text{if } Q \neq \emptyset \\ -1 & \text{otherwise,} \end{cases} \tag{3.7}$$

where $dist$ is the Euclidean distance between two points.

Definition 3.3.6. Let A be an approximation set for a MOP with Pareto optimal front PF. Given two elements $a_i, a_j \in A$, a_i contributes less than a_j to I_{Δ_p} if one of the following conditions holds:

- a_i contributes to I_{IGD} and a_j does not
  $C_{IGD}(a_i, A, PF) > 0$ and $C_{IGD}(a_j, A, PF) < 0$
- a_i contributes more to I_{IGD} than a_j
  $C_{IGD}(a_i, A, PF) > C_{IGD}(a_j, A, PF)$
- a_i and a_j contribute equally to I_{IGD} but a_i is closer to the Pareto optimal front
  $C_{IGD}(a_i, A, PF) = C_{IGD}(a_j, A, PF)$ and $C_{GD}(a_i, A, PF) < C_{GD}(a_j, A, PF)$

The complexity of Δ_p-DDE is ruled by the building of the reference set (see [9]), which is exponential with respect to the number of objectives.

Algorithm 6 Pseudocode of the Δ_p-DDE

Input: MOP, stopping criteria,
Output: Pareto set approximation

Initialize population P
Evaluate population P
Initialize reference set S
Compute the Δ_p contributions of each individual according to S
while stopping criteria is not fulfilled **do**
  Generate offspring P' using variation operators
  Evaluate population P'
  $P \leftarrow P \cup P'$
  Update reference set S
  Compute the Δ_p contributions of P according to S
  Remove the worst individuals from P
return P

3.4 The $R2$ Indicator

The family of R indicators was original proposed by Hansen and Jaszkiewicz in 1998 [167], for the purpose of comparing approximations of different multi-objective optimizers by means of utility functions, which map a vector $\vec{y} \in \mathbb{R}^m$ to a scalar utility value $u \in \mathbb{R}$.

Specifically, the $R2$ indicator is a recommended approach [28] for many-objective problems, since it is weakly monotonic and simultaneously evaluate all desired aspects of a Pareto front approximation, being often preferred over the hypervolume for two reasons: first, its lower computational cost, and second, the more uniform distributions produced. In the following, we derived its expression.

For a set U of general utility functions, a probability distribution p on U, and a reference set R, the $R2$ indicator of a solution set A is defined as the expected utility:

$$R2(R, A, U, p) = \int_{u \in U} \max_{\vec{r} \in R} \{u(\vec{r})\}\, p(u) du - \int_{u \in U} \max_{\vec{a} \in A} \{u(\vec{a})\}\, p(u) du. \tag{3.8}$$

For a discrete and finite set U and a uniform distribution p over U, the $R2$ indicator can be written as [185]:

$$R2(R, A, U) = \frac{1}{|U|} \sum_{u \in U} \left(\max_{\vec{r} \in R} \{u(\vec{r})\} - \max_{\vec{a} \in A} \{u(\vec{a})\} \right). \tag{3.9}$$

Since the first summand is a constant if we assume R to be constant, we delete the first summand and call the resulting unary indicator also $R2$ for simplicity [12]:

$$R2(A, U) = -\frac{1}{|U|} \sum_{u \in U} \max_{\vec{a} \in A} \{u(\vec{a})\}, \tag{3.10}$$

The $R2$ indicator is based on an assumption that we are allowed to add values of different utility functions. Therefore, it is also dependent on the scaling of the different utility functions. In this case, the lower the value of the measure is, the higher is the evaluation of A.

With respect to the choice of the utility functions u, there are several possibilities, such as, weighted sum, least squares, weighted Tchebycheff metric, etc. (see Chapter 4) These utility functions have an associated set of uniformly distributed weight vectors W and a reference point $\vec{z}$, in order to maintain diversity. The most common utility function is the weighted Tchebycheff metric:

$$R2(A : W, \vec{z^*}) = \frac{1}{|W|} \sum_{\vec{w} \in W} \min_{\vec{a} \in A} \left\{ \max_{i \in \{1,\ldots,m\}} w_i \left| a_i - z_i^* \right| \right\}. \quad (3.11)$$

The contribution of a solution $\vec{a} \in A$ to the $R2$ indicator is defined as:

$$C_{R2}(\vec{a}, A : W, \vec{z^*}) = R2(A : W, vecz^*) - R2(A \setminus \{\vec{a}\} : W, \vec{z^*}). \quad (3.12)$$

In Sections 3.4.1, 3.4.2 and 3.4.3 we provide three new algorithms which rely on the $R2$ indicator.

3.4.1 $R2$-EMOA

The $R2$ Evolutionary Multi-Objective Algorithm was proposed by Trautmann, et al. in 2013 [186]. This algorithm extends SMS-EMOA by replacing the hypervolume indicator with the unary $R2$ indicator. It is computationally less expensive than SMS-EMOA; however, it still uses dominance ranking as the main criterion in the selection mechanism.

Algorithm 7 shows the pseudocode of the $R2$-EMOA.[3] After a random initialization of the population (lines 1 and 2) and the generation of one new offspring per iteration (lines 3 to 6), the $R2$-EMOA uses standard nondominated sorting (see Algorithm 2), and deletes the solution with the worst rank that has the smallest $R2$ indicator value of the remaining population (lines 7 to 10). Ideal objective vector and the set of normalized weight vectors are direct parameters of the algorithm.

To optimize for speed, the weighted Tchebycheff value for each solution and weight vector is only calculated once and stored throughout the algorithm. Furthermore, instead of calculating the $R2$ value for all the solution sets $R_h \setminus \{x\}$, as in the pseudocode of Algorithm 7, the worst solution is determined within the set R_h directly (see [186]).

The overall complexity is ruled by the nondominated sorting of a population, $\mathcal{O}(N^2 m)$, where N represents the population size and m the number of objectives. The storage requirement is $\mathcal{O}(N^2)$.

$R2$-EMOA can deal with small populations for two and three objectives. Furthermore, decision makers' preferences can be included by adjusting the weight vector distributions of the indicator which results in a focused search behavior.

[3] The implementation of R2-EMOA is available in MATLAB for download at `http://inriadortmund.gforge.inria.fr/r2emoa`.

Algorithm 7 Pseudocode of the $R2$-EMOA

Input: MOP, stopping criteria, set of normalized weight vectors W, ideal objective vector $\vec{z^*}$
Output: Pareto set approximation

Initialize population P
Evaluate population P
while stopping criteria is not fulfilled **do**
  Generate offspring p from P using variation operators
  Evaluate individual p
  $P \leftarrow P \cup \{p\}$
  $\{R_1, \ldots, R_h\} \leftarrow$ nondominated-sort(P)
  $\forall x \in R_h : r(x) \leftarrow R2(R_h \setminus \{x\} : W, \vec{z^*})$
  $x^* \leftarrow \arg\min_{x \in R_h} r(x)$
  $P \leftarrow P \setminus \{x^*\}$
return P

3.4.2 $R2$-MOGA and $R2$-MODE

The $R2$ Multi-Objective Genetic Algorithm and $R2$ Multi-Objective Differential Evolution was proposed by Díaz-Manríquez et al. in 2013 [16]. These algorithms incorporate the $R2$ indicator to a modified version of the nondominated sorting method of NSGA-II (see Section 2.7.2), in order to separate individuals into layers. This approach is coupled to two different search engines, resulting in two MOEAs.

The set of weight vectors W is generated at each generation using the random design, described in Subsection 4.3.1 and the weighted Tchebycheff functions are assumed as utility functions. The reference point is updated per generation using the utopian point.

Algorithm 8 presents the generic pseudocode of these MOEAs. The general idea is to calculate the contributions of the entire population. Then, the individuals with a contribution greater than zero are isolated. These individuals will form the first layer. In the following, the contributions are computed to the remainder of the population (to those individuals that have a contribution of zero in the previous step), identifying and isolating again the individuals with a contribution greater than zero, and assigning them the second layer. This process is repeated until the entire population is segmented.

In the following, we deduce the complexity of this generic algorithm per generation. The computational cost of the weight vector generation is $\mathcal{O}(|W|m)$. The offspring generation is $\mathcal{O}(N)$. The reunion of parents and offspring is assumed to be constant, $\mathcal{O}(1)$, as well as the initialization of the next generation. The population evaluation and calculation of the reference point is made in $\mathcal{O}(Nm)$ each. Using additional storage, the contribution of each individual to a weight vector in line 13 can be done in $\mathcal{O}(N|W|m)$, assuming that the evaluation of the utility functions take $\mathcal{O}(m)$. In the worst case, where more than N individuals are in a layer, line 15 takes $\mathcal{O}(N)$. The set

Algorithm 8 Pseudocode of the $R2$-MOEA

Input: MOP, maximum number of generations G, population size N, number of normalized weight vectors $|W|$
Output: Pareto set approximation

1: $i \leftarrow 0$
2: Initialize population P_i
3: Evaluate population P_i
4: Calculate the reference point $\vec{z^*}$
5: **while** $i < G$ **do**
6: Create the set of W weight vectors at random of size $|W|$
7: Generate offspring P'_i using variation operators
8: Evaluate population P'_i
9: Update the reference point $\vec{z^*}$
10: $S \leftarrow P_i \cup P'_i$
11: $P_{i+1} \leftarrow \emptyset$
12: **while** $P_{i+1} < N$ **do**
13: $F \leftarrow \left\{p \in S \mid C_{R2}(p, S : W, \vec{z^*}) \neq 0\right\}$ (see eq. (3.12))
14: **if** $|P_{i+1}| + |F| > N$ **then**
15: Remove from F the worst $|P_{i+1}| + |F| - N$ candidates
16: $P_{i+1} \leftarrow P_{i+1} \cup F$
17: $S \leftarrow S \setminus F$
18: $i \leftarrow$i + 1
19: **return** P_i

operations in lines 16 and 17 are constant. Therefore, the overall complexity is ruled by the calculation of the $R2$ contributions, $\mathcal{O}(N^2|W|m)$. Additionally, $\mathcal{O}(|W|N + m(N + |W|))$ storage is required for this procedure.

The experiments made in [16] demonstrated that these pair of algorithms were competitive with NSGA-II, MOEA/D and SMS-EMOA for 2 and 3 objectives, and outperformed them in high dimensionality

3.4.3 $R2$-IBEA

The $R2$ Indicator Based Evolutionary Algorithm, which was proposed by Phan and Suzuki in 2013 [15] as an extension of IBEA (see Subsection 3.2.1), which eliminates Pareto dominance and performs a selection guided by a binary version of the $R2$ indicator. This binary version determines a superior-inferior relationship between two individuals (x and y) in objective space, and is given by the following equation:

$$I_{R2}(\vec{x}, \vec{y}) = R2(\{\vec{x}\} : W, \vec{z^*}) - R2(\{\vec{x} \cup \vec{y}\} : W, \vec{z^*}), \tag{3.13}$$

where each $R2$ value is obtained using equation (3.11). Thus, weighted Tchebycheff functions are assumed as utility functions.

Algorithm 9 Pseudocode of the $R2$-IBEA

Input: MOP, maximum number of generations G, population size N, set of normalized weight vectors W, constant κ
Output: Pareto set approximation
1: $i \leftarrow 0$
2: Initialize population P_i
3: Evaluate population P_i
4: Calculate the reference point $\vec{z^*}$
5: **while** $i < G$ **do**
6: Perform binary tournament selection
7: Generate offspring P_i' using variation operators
8: Evaluate population P_i'
9: Let be $P_{i+1} \leftarrow \{P_i \cup P_i'\}$
10: Update the reference point $\vec{z^*}$
11: Calculate the population fitness:
$(\forall p \in P_{i+1})\, p.fitness \leftarrow \sum_{q \in P_{i+1} \setminus \{p\}} -e^{-I_{R2}(q.\vec{f}, p.\vec{f}: W, \vec{z^*})/\kappa}$
12: **while** $|P_{i+1}| > N$ **do**
13: $p^* \leftarrow \arg\min_{p \in P_{i+1}} p.fitness$
14: $P_{i+1} \leftarrow P_{i+1} \setminus \{p^*\}$
15: Update the population fitness:
$(\forall p \in P_{i+1})\, p.fitness \leftarrow p.fitness + e^{-I_{R2}(p^*.\vec{f}, p.\vec{f}: W, \vec{z^*})/\kappa}$
16: $i \leftarrow$i + 1
17: **return** P_i

It is worth noting that if $\vec{x} \prec \vec{y}$, then $I_{R2}(\vec{x}, \vec{y}) = 0$; otherwise, $I_{R2}(\vec{x}, \vec{y}) \geq 0$, which satisfies the property of weak monotonicity:

- $I_{R2}(\vec{x}, \vec{y}) \leq I_{R2}(\vec{y}, \vec{x})$ if $x \prec y$
- $I_{R2}(\vec{x}, \vec{y}) \geq I_{R2}(\vec{y}, \vec{x})$ if $y \prec x$

The set of normalized weight vectors W is generated using the hypervolume approach, described in Subsection 4.3.3 of Chapter 4.

In Algorithm 9, we show the pseudocode of $R2$-IBEA, which is based on a Genetic Algorithm. In the first three lines, the initial population is generated at random, and then evaluated. In line 4, the reference point $\vec{z^*} = (z_1^*, \ldots, z_m^*)$ is calculated using equation (3.14):

$$z_i^* = \min_{p \in P} p.f_i - 2 \max_{j=1,\ldots,m} \left\{ \max_{p \in P} p.f_j - \min_{p \in P} p.f_j \right\} + \left(\max_{p \in P} p.f_i - \min_{p \in P} p.f_i \right), \quad (3.14)$$

where P represents the set of individuals in the current population, and we assume that each individual p is composed of a vector of m-objective functions $p.\vec{f}$.

In each ith-generation, the parents are chosen from the current population P_i with a binary tournament operator (line 6). Here, the relationship between individuals is determined by equation (3.13), preferring the superior one as a parent. If two individuals yield the same I_{R2} value, one of them is selected at random. In line 7 the selected parents produce the offspring population P_i' by means of the SBX (self-adaptative simulated binary crossover) and Polynomial mutation (see last part of Section 2.7). Then, the offspring is evaluated in each objective function in line 8.

In line 9, the offspring are combined with the parent population to form a candidates for the next generation P_{i+1}. From this resulting set, the reference point is updated using again equation (3.14) in line 10.

It is important to mention, that the location of the reference point is adjusted dynamically, according to the extent of the current solutions in objective space. Since the $R2$ indicator has an inherent bias towards the center of Pareto optimal fronts [12], $R2$-IBEA tends to correct this problem, placing the reference point far enough from the individuals (even in the infeasible region). Therefore, the density of weight vectors in extreme regions is increased, so that individuals in edge regions can be reached.

In line 11, the fitness of each individual ($p.fitness$) is calculated by applying the individual's I_{R2} value to an exponential amplification function. Here, a value of $\kappa = 0.005$ is recommended. Then, the worst individual (i.e., the one with the lowest fitness) is removed from P_{i+1} (lines 13 and 14). In line 15, fitness is recalculated for each of the remaining individuals in P_{i+1}. By repeating this removal process until $|P_{i+1}| = N$, the N individuals are selected to be used in the next generation.

In the following, we obtain the complexity of $R2$-IBEA per generation. The computational cost of the tournament selection is $\mathcal{O}(N)$, as well as the offspring generation. The combination of parents and offspring is constant $\mathcal{O}(1)$. The population evaluation and calculation of the reference point is made in $\mathcal{O}(Nm)$ each. In order to compute fitness and perform the reduction of the population in $\mathcal{O}(N^2)$, it is advisable to calculate first the $R2$ values for each pair of individuals. This can be done in $\mathcal{O}(N^2|W|m)$. Therefore, the overall complexity is ruled by the calculation of the $R2$ values, $\mathcal{O}(N^2|W|m)$. Additionally, $\mathcal{O}(N^2 + m(N + |W|))$ storage is required for this procedure.

The experimental results in [15] showed that $R2$-IBEA outperformed $R2$-EMOA, MOEA/D, NSGA-II and IBEA-ϵ with respect to optimality and diversity of the obtained solutions, for almost all instances of the ZDT and DTLZ test problems (see Appendix A) from two to five objectives.

3.5 Other Approaches

There are several other scalable techniques that can be applied to problems with more than three objectives as search engine into a MOEA. In the following sections, we briefly mention some of them, such as, scalarization of utility functions (Section 3.5.1), entropy (Section 3.5.2) and ϵ indicator (Section 3.5.3).

Algorithm 10 Pseudocode of the MOEA/D

Input: MOP, stopping criteria, set of normalized weight vectors: $(\forall p \in P)\, p.\vec{w}$, neighborhood size K, utility function u
Output: Pareto set approximation
1: Initialize population P
2: Evaluate population P
3: $(\forall p \in P)\, p.B \leftarrow$ k-nearest-neighbor(P, p, K)
4: Calculate the reference point $\vec{z^*}$
5: **while** stopping criteria is not fulfilled **do**
6:   **for all** $p \in P$ **do**
7:     Let a and b be two parents randomly selected from $p.B$
8:     Generate offspring c from a and b using genetic operators
9:     Evaluate individual c
10:     Update the reference point $\vec{z^*}$:

$$z_i \leftarrow \begin{cases} c.f_i & \text{if } c.f_i < z_i \\ z_i & \text{otherwise} \end{cases} \quad \forall i \in \{1, \ldots, m\}$$

11:     **for all** $q \in p.B$ **do**
12:       **if** $u(c.\vec{f} : \vec{z^*}, q.\vec{w}) \leq u(q.\vec{f} : \vec{z^*}, q.\vec{w})$ **then**
13:         Replace individual q by c
14: **return** P

3.5.1 MOEA/D

The Multi-Objective Evolutionary Algorithm based on Decomposition, proposed by Zhang in 2006 [14], transforms an optimization problem into a number of scalar optimization subproblems that are simultaneously optimized by evolving a population. Each individual solution in the population is associated with a subproblem. These subproblems are quantified using a utility function with uniformly distributed weight vectors and a reference point.

The set of normalized weight vectors W is generated using the simplex-lattice design approach, described in Subsection 4.3.2.

In Algorithm 10, we present the pseudocode of MOEA/D. Here, P represents the population and every individual p is associated with a weight vector, $p.\vec{w}$ and a neighborhood of this weight vector, $p.B$, which is defined as a set of its K closest weight vectors. The element $p.\vec{f}$ represents the m-objective functions.

In the first two lines, the population is initialized and evaluated. In line 3, the Euclidean distances are computed between any two weight vectors and then the K closest weight vectors for each individual p are selected and stored in record $p.B$. It is worth to mention that p itself belongs to $p.B$. In line 4, it is initialized the reference point with the best value found so far for each objective. At each generation, in lines 5 to 9, for every individual in the population, two neighbors are selected for generating and evaluating a new individual. The reference point is updated in line 10. Then, the new individual is compared with the neighbors of p, if it surpasses one of them, then it is replaced (lines 11 to 13). When the stopping criteria is fulfilled, it is return the final population, in line 14.

The storage required for MOEA/D is $\mathcal{O}(Nm)$, where $N = |P|$. The complexity of this procedure at each generation is $\mathcal{O}(NKm)$, assuming a cost of $\mathcal{O}(m)$ for the computation of the utility function.

MOEA/D has the lowest complexity of all the described MOEAs. In several studies, it have been shown that MOEA/D is able to outperform NSGA-II [187, 188].

3.5.2 Entropy

This is a performance indicator for spread based on Shanon's entropy, which has been proposed in [189]. The basic idea is that each solution point provides some information about its neighborhood modeled by a Gaussian distribution. A density function has also been calculated by the sum of all Gaussian distributions from all solution points. The peaks and valleys of density function correspond to the dense areas and the sparse areas, respectively. A desirable solution set should have a "uniform" density function which was evaluated with Shannon's entropy [189].

Originally, the term "flat" was used for the description of the density function, however, statistically the term "uniform" seems to be more appropriate.

3.5.3 Epsilon Indicator

The ϵ indicator, proposed in [10], is the minimum factor for which we need to multiply all the elements of a reference set P in order to have all its elements dominated or equal to the elements in a nondominated set A. Smaller values of this indicator means that the set A is more similar to the reference set and that it is a better NS (because the reference set is suppose to be a better NS than any set in the comparison). It is compatible with all the outperformance relations. It is easy to calculate and has a low computational complexity .

3.6 Summary

In this chapter we reviewed three important performance indicators that are suitable as guidance for the search during the optimization process.

The hypervolume indicator is the only quality indicator known to be fully sensitive to Pareto dominance and is invariant to scaling of objectives. Nevertheless, its high computational cost (it grows exponentially on the number of objectives [8]) normally makes a selection mechanism based on such indicator prohibitive for problems having more than 5 objectives [9]. Another disadvantage is that it is sensitive to the choice of the reference point. Examples of algorithms are: IBEA and SMS-EMOA.

Another indicator is Δ_p, which can be seen as an "averaged Hausdorff distance" between the outcome set and the Pareto optimal front. It is composed of slight modifications of the GD and IGD indicators. Even though Δ_p is not Pareto compliant, it evaluates proximity to the Pareto optimal front and spread of solutions along it. Moreover, it has a much lower computational cost than that of the hypervolume, and it can also handle outliers. The representative algorithms are: Δ_p-EMOA and Δ_p-DDE.

The $R2$ indicator is only weakly monotonic and is variant to scaling, but it has a correlated behavior with the hypervolume. Furthermore, this indicator has a low computational cost and the distribution obtained using this indicator are more uniform than those of the hypervolume. Examples of algorithms are: $R2$-EMOA, $R2$-MOGA, $R2$-MODE and $R2$-IBEA.

Finally, there exist some other approaches for incorporating into a MOEA, such as scalarization, entropy and ϵ indicator.

Chapter 4

A New Metaheuristic for Many-Objective Optimization

This chapter presents our proposed MOEA, called Many-Objective Metaheuristic Based on the $R2$ Indicator (MOMBI), which performs a hierarchical ranking of the individuals of a population. The ranking procedure is described in Section 4.1. As we saw in Chapter 3, the $R2$ indicator requires a set of utility functions that measures the decision maker's relative preference for the solutions. In Section 4.2, we detail the most effective choices of these utility functions. Since MOEAs are considered a posteriori methods, the set of available utility functions must give a wide range of possibilities over the whole objective space. In Section 4.3, we describe four methods intended to produce distributed weight vectors. In Section 4.4, we integrate the metaheuristic and provide a hypothetical example. Then, in Section 4.5, we suggest an interactive method for MOMBI in which the decision maker can produce individuals around specific reference points. The source code of MOMBI is available for download at: `http://computacion.cs.cinvestav.mx/~rhernandez/mombi`. Finally, in Section 4.6 we provide a summary of the chapter.

4.1 The Proposed Ranking Algorithm

According to Brockhoff et al. [12], the unary version of the $R2$ indicator for a constant reference set can be expressed as follows:

$$R2(A, U) = \frac{1}{|U|} \sum_{u \in U} u^*(A), \tag{4.1}$$

where A is the Pareto set approximation, U is a set of utility functions and $u^*(A) = \min_{\vec{a} \in A}\{u(\vec{a})\}$ is the best utility value obtained in the set A.[1]

Since we intend to use $R2$ in the selection mechanism of a MOEA, we need to design a scheme for that purpose. Our proposal here is to produce a nondominated

[1]For simplicity, we have applied the dual property: $\min \vec{z} = -\max(-\vec{z})$, and we also assume that the utility functions only take positive values.

sorting scheme based on the utility functions adopted. The core idea is to group solutions that optimize the set of utility functions chosen, and place these solutions on top, such that they get the first rank (the best). Such points will then be removed and a second rank will be identified in the same manner. The process will continue until all the solutions had been ranked. Clearly, this is a nondominated sorting scheme proposed by Goldberg [78], except for the fact that Pareto dominance is not used in this case.

The formal definition of a rank, derived from equation (4.1), is presented in equation (4.2):

$$\text{rank}_k(A) = \bigcup_{\vec{u} \in U} u^*(\{A \setminus B_k(A)\}), \tag{4.2}$$

where $B_k(A)$ is the union of solutions with the lowest ranks:

$$B_k(A) = \left\{ \bigcup_j \text{rank}_j(A) | k > 1, j \in [1, k) \right\}. \tag{4.3}$$

When two individuals contribute with the same utility value, then we propose to choose as tiebreaker the one with the lower L_2 and L_1 norms [190] (also known as Euclidean and Manhattan norms respectively). These norms are defined by:

$$L_p(\vec{x}) = \left(\sum_{i=1}^{m} |x_i|^p \right)^{1/p}, \tag{4.4}$$

where m is the number of objectives.

In Algorithm 11, we present a naive approach to rank a population, based on equations (4.2) and (4.3). We assume that each individual p has the following structure:

- $p.\vec{f}$: Vector of objective function values
- $p.L_1$: Manhattan norm of $p.\vec{f}$
- $p.L_2$: Euclidean norm of $p.\vec{f}$
- $p.rank$: Hierarchy of the individual
- $p.u^*$: The best utility value obtained
- $p.\alpha$: The current utility value for a utility function u

In lines 1 to 5, for every pair of objective vector $\vec{f}$ and utility function u of an individual p, the current utility value is computed and stored in record $p.\alpha$. If the obtained value of an individual outperforms its previous one, it is updated in field $p.u^*$. In line 6, the population P is sorted with respect to the Algorithm 12. Lines 7 to 11 perform the ranking assignment of the sorted population.

Algorithm 11 $R2$ Ranking Algorithm

Input: Population P, set of utility functions U
Output: Ranking of the population

for all $\vec{u} \in U$ **do**
  for all $p \in P$ **do**
    $p.\alpha \leftarrow u(p.\vec{f})$
    if $p.\alpha < p.u^*$ **then**
      $p.u^* \leftarrow p.\alpha$
  Sort the population P using Algorithm 12 in increasing order.
  $rank \leftarrow 1$
  for all $p \in P$ **do**
    if $rank < p.rank$ **then**
      $p.rank \leftarrow rank$
    $rank \leftarrow rank + 1$

As mentioned before, when two individuals have the same utility value, we prefer the solution with the lowest norms, since this guarantees convergence towards the Pareto optimal front.

In the following, we obtain the complexity of the ranking algorithm. First, assuming that the evaluation of a utility function requires $\mathcal{O}(m)$, the computation of a utility function for all the individuals takes $\mathcal{O}(|P|m)$, where $|P|$ denotes the population size. The complexity of comparing individuals is constant; therefore, the sorting procedure is performed in $\mathcal{O}(|P| \log |P|)$. Moreover, the ranking assignment is linear with respect to the population, $\mathcal{O}(|P|)$. Since these steps are computed $|U|$ times, the overall complexity is $\mathcal{O}(|U||P|(\log |P| + m))$. Furthermore, it is worth noting that $\mathcal{O}(|P|m)$ storage is required for this procedure.

The choice of the utility functions becomes crucial to achieve a ranking compatible with Pareto dominance. In the next section we will focus on this issue.

4.2 Utility Functions

The concept of utility was developed in the early 1940s by Von Neumann and Morgenstern in the axiomatic utility theory [191], which assumes that the decision maker can choose among the alternatives available in such a way that the satisfaction derived from the choice made is as large as possible. This, of course, implies the decision maker is aware of the alternatives available and is capable of evaluating them. Moreover, it is assumed that all information pertaining to the various levels of the objectives can be captured by a utility function.

Mathematically, a utility function, $u : \mathbb{R}^m \to \mathbb{R}$, is a model of the decision maker's preference that maps each point in the objective space into a utility value. It is important to mention that a utility function does not really reflect the decision maker's inner (psychological) intensity of preference. It just provides a model of

Algorithm 12 Comparison Between Individuals

Input: Individuals a and b
Output: The best individual

```
if a.α < b.α then
    return a
if a.α > b.α then
    return b
if a.u* < b.u* then
    return a
if a.u* > b.u* then
    return b
if a.L2 < b.L2 then
    return a
if a.L2 > b.L2 then
    return b
if a.L1 < b.L1 then
    return a
if a.L1 > b.L1 then
    return b
return "tie"
```

his behavior [192]. This is an important distinction, since behavior should then be consistent (i.e., it should not originate intransitivities[2]).

The weighted L_p metrics and some variants that are based on optimization techniques are the most suitable for representing utility functions. In the following subsections we explore the most popular metrics for generating Pareto optimal solutions.

4.2.1 Weighted L_p Metrics

The L_p metrics (see equation (4.4)) can be weighted in order to produce different (weakly) Pareto optimal solutions. The *weighted L_p metric* for $p \in [1, \infty)$ is defined as:

$$u_{L_p}(\vec{a} : \vec{r}, \vec{w}) = \left(\sum_{i=1}^{m} w_i |a_i - r_i|^p \right)^{1/p}, \tag{4.5}$$

where $\vec{r} \in \mathbb{R}^m$ is a reference point, $\vec{w}$ is a weight vector such that $w_i \geq 0$ for all $i = 1, ..., m$ and $\sum_{i=1}^{m} w_i = 1$.

If $p = 1$, the metric is better known as *weighted sum function*. The minimization of this metric is always Pareto optimal if the weighting coefficients are all positive or if the solution is unique; the corresponding proof can be found in [32]. The weakness

[2]Let a, b and c be three solutions, if $u(a) < u(b)$ and $u(b) < u(c)$, then it is impossible that $u(c) < u(a)$.

of the weighted sum function is that not all of the Pareto optimal solutions can be found unless the problem is convex. The same weakness may also occur in problems with discontinuous objective functions [193]. In Figure 4.1 (a)-(b) the contour lines of this metric are shown for a bi-objective problem. The bold curve represents the Pareto optimal front, the dashed line is the weight vector and the black points are the optimal solutions, assuming the reference point is placed at the origin. The darker the contour, the better the utility is. It is worth noting that only extreme solutions can be found with this technique.

If $p = 2$, we have the *metric of least squares*, whose representation is illustrated in Figure 4.1 (c)-(d). It is important to mention that even though the minimization of the weighted L_p-metrics ($1 <= p < \infty$) produces Pareto optimal solutions, it does not guarantee to find all of them.

Finally, when $p = \infty$, the metric is also called *weighted Tchebycheff metric*, and it is of the form:

$$u_{L_\infty}(\vec{a} : \vec{r}, \vec{w}) = \max_{i=1,\dots,m} \{w_i |a_i - r_i|\}\,. \tag{4.6}$$

It has been shown that the minimization of the weighted Tchebycheff metric is weakly Pareto optimal, if all the weighting coefficients are positive [32]. Thus, an auxiliary calculation is needed in order to identify weak solutions. The geometric interpretation of equation (4.6) is presented in Figure 4.1 (e)-(f).

4.2.2 Variants of the L_∞ Metric

Weakly Pareto optimal solutions can be avoided by giving a slight slope to the contour of the metric. The price to be paid is that in some cases it may be impossible to find every Pareto optimal solution [32]. It is suggested in Steuer [194] and Steuer and Choo [195] that the weighted Tchebycheff problem can be varied by an augmentation term. This metric is named *augmented weighted Tchebycheff metric* and it is of the form:

$$u_{aug}(\vec{a} : \vec{r}, \vec{w}) = \max_{i=1,\dots,m} \{w_i |a_i - r_i|\} + \rho \sum_{j=1}^{m} |a_j - r_j|, \tag{4.7}$$

where ρ is a sufficiently small positive scalar (see Figure 4.2).

A slightly different modified weighted Tchebycheff metric is used in the *modified weighted Tchebycheff metric*:

$$u_{mod}(\vec{a} : \vec{r}, \vec{w}) = \max_{i=1,\dots,m} \left\{ w_i \left(|a_i - r_i| + \rho \sum_{j=1}^{m} |a_j - r_j| \right) \right\}. \tag{4.8}$$

The difference between the augmented and the modified weighted Tchebycheff metrics is in the way the slope takes place in the metrics. In the augmented weighted

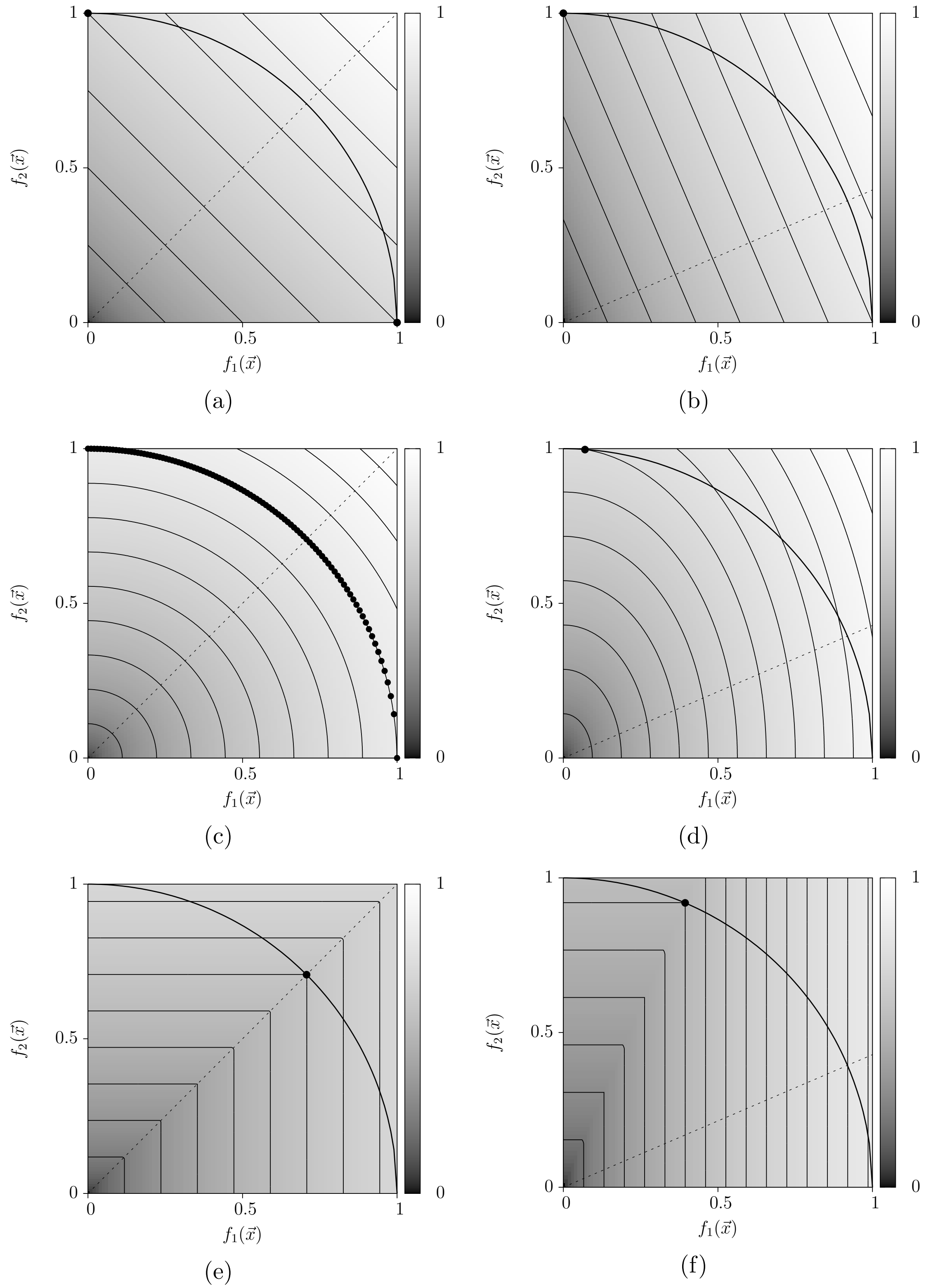
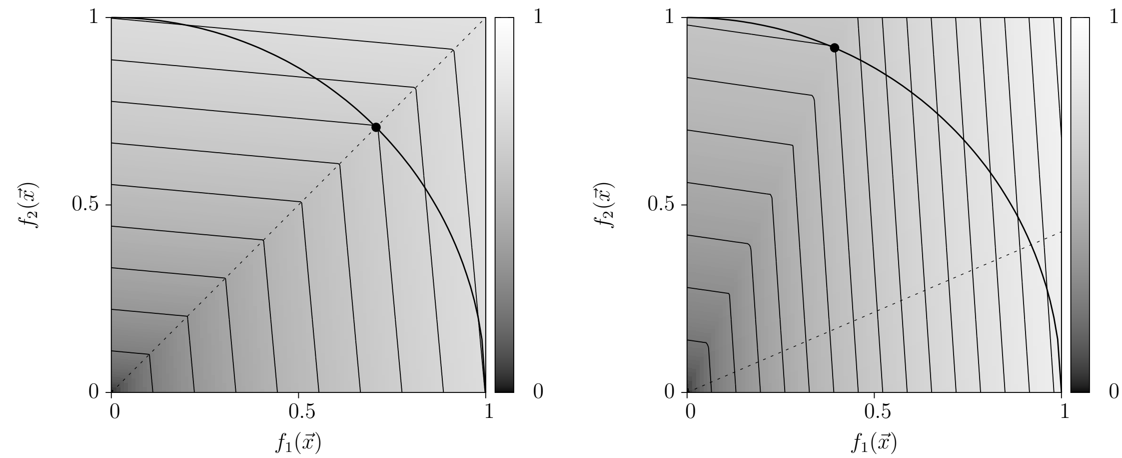


Figure 4.1: Contour lines of the L_p metrics for the weight vectors $\vec{w} = (0.5, 0.5)$ and $\vec{w} = (0.7, 0.3)$. $p = 1$: (a)-(b), $p = 2$: (c)-(d) and $p = \infty$: (e)-(f).



(a) $\vec{w} = (0.5, 0.5)$ (b) $\vec{w} = (0.7, 0.3)$

Figure 4.2: Contour lines of the augmented weighted Tchebycheff metric.

Tchebycheff problem the slope is a function of the weighting coefficients and the parameter ρ.[3] In other words, the slope may be different for each objective function. As far as the modified weighted Tchebycheff metric is concerned, the slope is a function of the parameter ρ and, thus, constant for all the objective functions.[4]

When objectives are incommensurable, it is advisable to rescale or normalize the objective functions, so that their objective values are of approximately the same magnitude. In this case, the expression $|a_k - r_k|$ of the utility functions above described is replaced by:

$$\frac{a_k - z_k^*}{z_k^{nad} - z_k^*}, \quad \forall k \in \{1, \cdots, m\} \tag{4.9}$$

where $\vec{z}^*$ and $\vec{z}^{nad}$ are the ideal and nadir objective vectors, respectively (see Definitions 2.3.3 and 2.3.1). With this modification, the range of each new objective function is $[0, 1]$.

4.2.3 Penalty-based Boundary Intersection

The *Penalty-based Boundary Intersection (PBI) metric*, originally proposed by Zhang and Li [14], is a modification of the Normal Boundary Intersections (NBI) [196], and it is of the form:

$$u_{pbi}(\vec{a} : \vec{r}, \vec{w}) = d_1 + \theta d_2, \tag{4.10}$$

$$d_1 = \frac{\left|\left|(\vec{a} - \vec{r})^T \vec{w}\right|\right|}{||\vec{w}||}, \tag{4.11}$$

[3] $\beta_i = \arctan \frac{\rho}{1 - w_i + \rho}$
[4] $\beta = \arctan \frac{\rho}{1 + \rho}$

$$d_2 = \left|\left|\vec{a} - \left(\vec{r} + d_1 \frac{\vec{w}}{||\vec{w}||}\right)\right|\right|. \quad (4.12)$$

where $\theta > 0$ is a penalty parameter.

The minimization of the PBI technique produces Pareto optimal solutions much more uniformly distributed than those obtained by the weighted L_∞ metric. Additionally, this approach is able to deal with concave Pareto optimal fronts. A too large or too small penalty factor will worsen the performance of the penalty method, and, therefore, a value of $\theta = 5$ is recommended [14]. This method produces results similar to those produced by the weighted sum function (when $\theta = 0$), and the weighted Tchebycheff metric (when $\theta = 1$). In the second case, the axis are rotated. This is depicted in Figure 4.3.

It is important to mention that all the described utility functions can be implemented in $\mathcal{O}(m)$ time complexity.

4.3 Weights

The utility functions require a specified number of weight vectors $\vec{w} = (w_1, ..., w_m)$, such that $w_i \geq 0$ for all $i = 1, ..., m$ and $\sum_{i=1}^{m} w_i = 1$. These vectors describe a simplex surface, and must be uniformly distributed across objective space in order to obtain different solutions of the Pareto optimal front. In this section, we will present four different ways in which they can be generated. This way, the decision maker avoids the difficult task of specifying the coefficient values, and only generates and stores once these samples, for then integrating them into a metaheuristic.

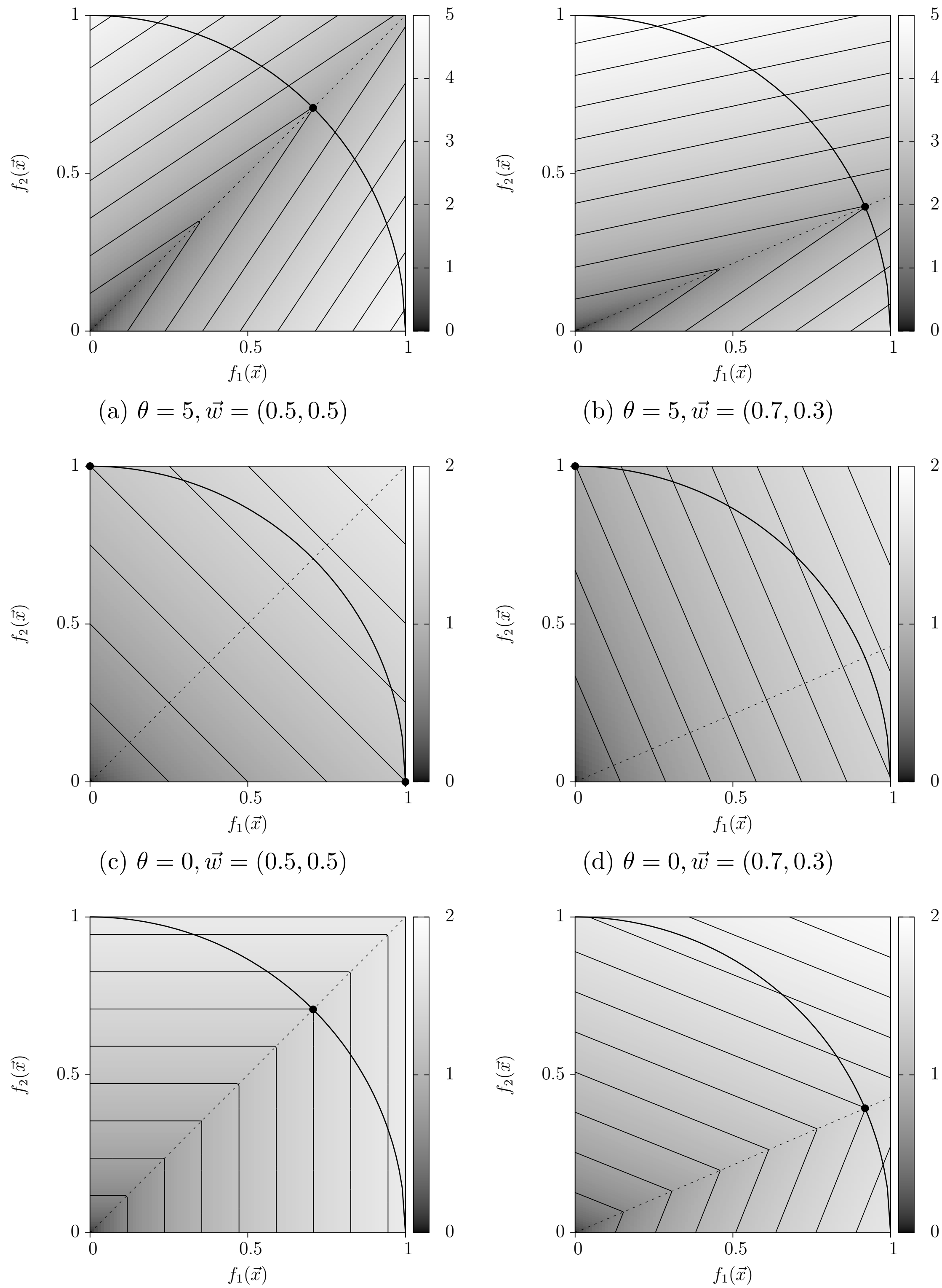
4.3.1 Randomized Design

In this approach, the normalized weight vectors are drawn at random. Algorithm 13, proposed by Steuer in 1986 (see [194]), uniformly samples the whole simplex space.

Algorithm 13 Random Design by Steuer

Input: Dimension m, number of weight vectors l
Output: Set of normalized weight vectors

```
W ← ∅
while |W| < l do
    ∀i ∈ {1, ..., m}. w_i ← rand(0, 1)
    s ← Σ_{i=1}^{m} w_i
    if s ≠ 1 then
        w⃗ ← w⃗/s
    W ← W ∪ {w⃗}
return W
```



(a) $\theta = 5, \vec{w} = (0.5, 0.5)$ (b) $\theta = 5, \vec{w} = (0.7, 0.3)$

(c) $\theta = 0, \vec{w} = (0.5, 0.5)$ (d) $\theta = 0, \vec{w} = (0.7, 0.3)$

(e) $\theta = 1, \vec{w} = (0.5, 0.5)$ (f) $\theta = 1, \vec{w} = (0.7, 0.3)$

Figure 4.3: Contour lines of the PBI approach.

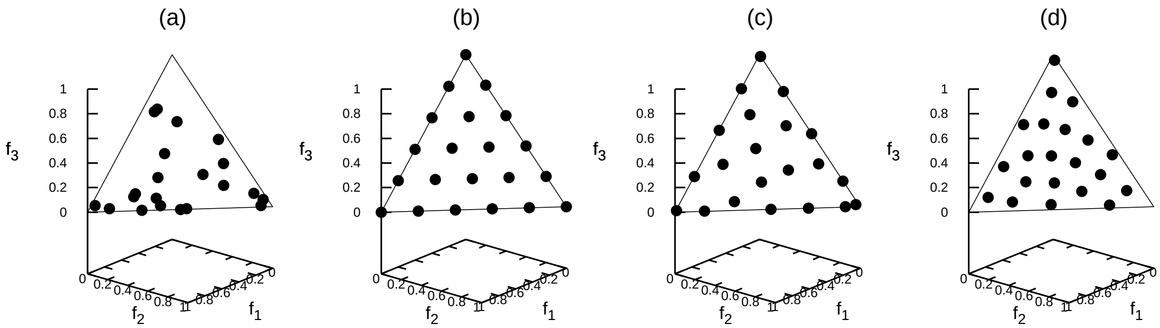


Figure 4.4: Weight vector generation for $m = 3$: (a) randomized design, (b) simplex-lattice design, (c) hypervolume-based design and (d) uniform design.

Here, $\vec{w} \in \mathbb{R}^m$ is a weight vector, and the function rand$(0, 1)$ returns a random value within the range $(0, 1]$ with uniform probability.

Another variant, proposed by Jaszkiewicz in 2002 (see [197]), is described by the following expression:

$$
\begin{aligned}
w_1 &= 1 - \sqrt[m-1]{\text{rand}(0,1)} \\
&\vdots \\
w_i &= \left(1 - \textstyle\sum_{j=1}^{i-1} w_j\right)\left(1 - \sqrt[m-1-i]{\text{rand}(0,1)}\right) \\
&\vdots \\
w_m &= 1 - \textstyle\sum_{j=1}^{m-1} w_j,
\end{aligned}
\tag{4.13}
$$

Random design is easy to implement; however, the distribution produced of the Pareto optimal front is poor. An example for 3D is shown in Figure 4.4-(a).

4.3.2 The Simplex-Lattice Design

This method, introduced by Scheffé in 1958-1965 (see [198], [199], [200]), is the foundation on which the theory of experimental designs for mixtures was built, and these designs are still in use today. In this case, the distribution of the weight vectors are equally spaced over the simplex, forming an ordered arrangement called $\{m, h\}$ simplex-lattice, where m is the number of objectives and h is a parameter of proportion. This structure consists of all possible combinations of proportions of each objective function, thus every weight coefficient takes $h + 1$ equally spaced values from 0 to 1, that is:

$$w_i = \varepsilon, \frac{1}{h}, \frac{2}{h}, ..., 1 \tag{4.14}$$

where ε is a value close to zero (10^{-4} is recommended), in order to prevent cancellation in subsequent calculations. The number of weight vectors in the $\{m, h\}$ simplex-lattice

Algorithm 14 Simplex-Lattice Design

Input: Dimension m, parameter of proportion h
Output: Set of normalized weight vectors

Initialize vector $\vec{x} = (x_1, ..., x_m)$, such that:

$$x_i \leftarrow \begin{cases} h & \text{if } i = 1 \\ 0 & \text{otherwise} \end{cases}$$

$W \leftarrow \emptyset$
$j \leftarrow 1$ {Target element}
repeat
  $W \leftarrow W \cup \left\{ \frac{\vec{x}}{h} \right\}$
  $x_j \leftarrow x_j - 1$
  if $j < m - 1$ **then**
    $x_{j+1} \leftarrow h - \sum_{i=1}^{j} x_j$
    $\forall i \in \{j+2, ..., m\}.\ x_i \leftarrow 0$
    $j \leftarrow j + 1$
  else
    $x_m \leftarrow x_m + 1$
    $I \leftarrow \{i | x_i > 0, i = 1, ..., m-1\}$
    if $I \neq \emptyset$ **then**
      $j \leftarrow max(I)$
until $x_m = h$
$W \leftarrow W \cup \left\{ \frac{\vec{x}}{h} \right\}$
return W

is given by the combinatorial number[5] $C_{m-1}^{h+m-1} = \frac{(h+m-1)!}{h!(m-1)!}$.

There are several proposed algorithms that implement the simplex-lattice design. In Algorithm 14, we present an efficient one described by Chasalow and Brand in 1995 [201].

The disadvantage of the simplex-lattice design is that the points are not uniformly distributed over the domain, because if there are too many points at the boundary, it produces too many insignificant weight values, and, in consequence, the quality of the Pareto optimal solutions is affected. In addition, this approach is not scalable, since the number of weight vectors is restricted to a combinatorial number, which becomes impractical in evolutionary algorithms when the population size is related to the number of weight vectors. See Figure 4.4-(b) for an example.

[5] C_b^a is the combinatorial symbol for the number of ways a things can be taken b at a time and $C_b^a = \frac{a!}{b!(a-b)!}$.

Algorithm 15 Hypervolume-Based Design by Phan

Input: Dimension m, number of weight vectors l, maximum number of iterations T, reference point $\vec{z}$

Output: Set of normalized weight vectors

1: $t \leftarrow 0$
2: $W \leftarrow \emptyset$
3: **while** $t < T$ **do**
4: Create a random vector $\vec{x}$: $\forall i \in \{1, \ldots, m-1\}.\ x_i \leftarrow rand(0, 1)$
5: Sort $\vec{x}$ in increasing order
6: $\vec{w} \leftarrow (x_1, x_2 - x_1, \ldots, x_{m-1} - x_{m-2}, 1 - x_{m-1})$
7: $W \leftarrow W \cup \{\vec{w}\}$
8: **if** $|W| > l$ **then**
9: $\vec{w}^* \leftarrow \arg\min_{\vec{w} \in W} C_{HV}(\vec{w}, W : \vec{z})$ (see eq. (3.2))
10: $W \leftarrow W \setminus \{\vec{w}^*\}$
11: $t \leftarrow t + 1$
12: **return** W

4.3.3 Hypervolume-Based Design

This design, introduced by Phan in 2013 (see [15]), generates distributed weight vectors using an approach similar to SMS-EMOA (see Subsection 3.2.2). In this case, the hypervolume indicator is used for quantifying the distribution of the weight vectors in objective space, normalized into $[0, 1]$.

Algorithm 15 presents the pseudocode of this design. In lines 1 and 2, it is initialized the number of generations and the set of weight vectors. At each iteration, a vector $\vec{x}$ is randomly chosen from $[0, 1]^{m-1}$, following a uniform distribution (line 4). In lines 5 and 6, an m-dimensional vector w is created by sorting $\vec{x}$. This vector is added to the set W in line 7. If the number of weight vectors is greater than the allowed vectors, then the hypervolume contributions are calculated for all the weight vectors, removing the one that contributes less to the hypervolume indicator (lines 8 to 10). At the end, the set W is returned.

This method ensures maximizing the hypervolume indicator of the solutions produced, but at a high computational cost. The computational complexity at each iteration is $\mathcal{O}(l^m)$, assuming the hypervolume algorithm described by Zitzler (see Table 3.1). Figure 4.4-(c) shows the weight vectors generated by using this method with $m = 3$, $l = 21$, $T = 5000$ and the reference point is at $(2, 2)$.

4.3.4 Uniform Design

Uniform design was proposed by Fang and Wang [202, 203] and has been applied in many areas since 1980, such as industry, systems engineering, chemistry, pharmaceutics, natural sciences, etc. Its purpose is to look for points uniformly scattered on the experimental domain and predict the response when the underlying model is unknown. The incorporation of this approach in the weight vector generation for MOEAs was first proposed by Tan, et al. in 2012 (see [204]). We will next provide some basic definitions related to uniform design.

Definition 4.1. A U-type design[6] denoted by $\mathbf{U} : \mathcal{G} \subset \mathbb{Z}_{>0}^{l\times n}$ is a matrix of l rows and n columns, such that each column is a permutation of the entries $\{1, ..., l\}$.

Note that many of the elements of $\mathcal{G}$ (the set of all U-type designs) may have a poor uniformity.

Definition 4.2. A transformation $T : \mathbb{Z}_{>0}^{l\times n} \to [0,1]^{l\times n}$ of a U-type design $\mathbf{U}$ to the n-dimensional unit cube, is expressed by $T(\mathbf{U}) = \mathbf{C} = (c_{ij})$, where:

$$c_{ij} = \frac{(u_{ij} - 0.5)}{l} \text{ for all } i = 1, ..., l \text{ and } j = 1, ..., n. \tag{4.15}$$

Definition 4.3. Let $M : [0,1]^{l\times n} \to \mathbb{R}$ be a measure of uniformity such that the smaller value of M, the better the uniformity of the design.

Definition 4.4. $\mathbf{U} \in \mathcal{G}$ is called a uniform design under the measure M if:

$$M(T(\mathbf{U})) = \min_{\mathbf{V}\in\mathcal{G}} M(T(\mathbf{V})). \tag{4.16}$$

The most popular measures of uniformity rely on quasi-Monte-Carlo methods, such as the L_p-discrepancy [205, 206]. This measure is invariant to the permutation of rows and columns. However it is expensive to compute, thus attempts have been made to evaluate the discrepancy algorithmically.

One approach is the centered L_2-discrepancy (CD for short), it is also invariant under coordinate rotations. Hickernell [207] gave an analytical expression for the CD:

$$\begin{aligned}(CD(\mathbf{C}))^2 = &\left(\frac{13}{12}\right)^n - \frac{2}{l}\sum_{k=1}^{l}\prod_{j=1}^{n}\left(1 + \frac{1}{2}|c_{kj} - 0.5| - \frac{1}{2}|c_{kj} - 0.5|^2\right) \\ &+ \frac{1}{l^2}\sum_{k=1}^{l}\sum_{j=1}^{l}\sum_{i=1}^{n}\left[1 + \frac{1}{2}|c_{ki} - 0.5| + \frac{1}{2}|c_{ji} - 0.5| - \frac{1}{2}|c_{ki} - c_{ji}|\right].\end{aligned} \tag{4.17}$$

where $\mathbf{C} = (c_{ki})$ is a $l \times n$ matrix.

The solution of the combinatorial optimization problem in (4.16) is not unique when $n > 1$. Additionally, the search for uniform designs is an NP hard problem, as l and n increase [208]. Due to this, there are several methods that can provide a good approximation to the uniform design, such as the good lattice method, Latin square method, expending orthogonal design method, and optimization searching [208].

In the remainder of this section, we will focus on finding a nearly uniform design of the set of weight vectors W (for simplicity $l = |W|$ and m is the number of objective

[6] U stands for uniform design.

Algorithm 16 Uniform Design + Tabu Search

Input: Dimension m, number of weight vectors l, maximum number of iterations T, neighborhood size S

Output: Set of normalized weight vectors

1: Find the candidate set of positive integers:

$$H_l \leftarrow \{h \in \mathbb{Z}_{>0} | h < l, gcd(l, h) = 1\}.$$

2: Create a random vector:

$$\vec{x} \leftarrow \{(x_1, ..., x_{m-1}) \mid 0 < i < m, x_i \in H_l\}.$$

3: $\vec{x}^* \leftarrow \vec{x}$
4: $L \leftarrow \{\vec{x}\}$
5: $t \leftarrow 0$
6: **while** $t < T$ **do**
7: $\quad N \leftarrow \text{neighborhood}(\vec{x}, m-1, H_l, S) \setminus L$
8: $\quad$ **if** $N = \emptyset$ **then**
9: $\quad\quad L \leftarrow L \cup \{\vec{x}\}$
10: $\quad\quad$ Execute step 2
11: $\quad$ **else**
12: $\quad\quad \vec{x} \leftarrow \arg\min_{\vec{y} \in N} CD(U\text{-type}(\vec{y}, l, m-1))$ (see eq. (4.17))
13: $\quad\quad L \leftarrow L \cup (N \setminus \{\vec{x}\})$
14: $\quad\quad$ **if** $\vec{x}$ is better than or equal to $\vec{x}^*$ **then**
15: $\quad\quad\quad \vec{x}^* \leftarrow \vec{x}$
16: $\quad t \leftarrow t + 1$
17: Build $l \times (m-1)$ matrix:

$$(c_{ij}) \leftarrow U\text{-type}(\vec{x}^*, l, m-1)$$

18: Generate the set of weight vectors $W = \{\vec{w}_k = (w_{k1}, ..., w_{km}) \mid k = 1, ..., l\}$, such that:

$$\begin{aligned} w_{ki} &= \left(1 - c_{ki}^{\frac{1}{m-i}}\right) \prod_{j=1}^{i-1} c_{kj}^{\frac{1}{m-j}}, i = 1, ..., m-1, \\ w_{km} &= \prod_{j=1}^{m-1} c_{kj}^{\frac{1}{m-j}} \end{aligned}$$

19: **return** W

functions) using our proposed technique, which is based on Tabu search and the good lattice point method.[7]

In Algorithm 16 we present our proposal. In step 1, the candidate set of positive

[7] The good lattice point method is an efficient quasi Monte Carlo method, proposed by Korobov in 1959 [209].

Algorithm 17 U-type

Input: Vector $\vec{x}$, number of weight vectors l, dimension n
Output: U-type matrix of l rows and n columns

Generate $l \times n$ matrix $\mathbf{U} \leftarrow (u_{ij})$, where:

$$u_{ij} = (ix_j) \bmod l,$$

and the multiplication operation modulo l is modified as $1 \leq u_{ij} \leq l$.

Apply transformation $\mathbf{U} \leftarrow T(\mathbf{U})$ using eq. (4.15)

return $\mathbf{U}$

integers is generated by finding the co-prime numbers of l. Here, *gcd* corresponds to the greatest common divisor. In line 2, the vector $\vec{x}$ is generated from any $m-1$ distinct elements of H_l. Then, in lines 3 and 4, the best solution found so far $\vec{x}^*$ and the Tabu list L are initialized using $\vec{x}$. At each generation, in line 7, a set of candidate solutions N is generated by exchanging one element of $\vec{x}$ by one in H_l. These solutions must not be included in the Tabu list and it is advisable that the elements in each vector must be sorted. The cardinality of this set is at most S. If the set N is empty, then the solution $\vec{x}$ is added to the Tabu list and a new solution is generated (lines 8 to 10). Otherwise, in line 12, the best candidate solution from N is found by applying the measure CD on a U-type matrix (see Algorithm 17). In line 13, the Tabu list is updated, incorporating the worst solutions of N. In lines 14 and 15, if the best candidate solution outperforms the current best solution, then it is updated. At the end of the iterations, the U-type matrix is build using $\vec{x}^*$ (line 17), and from this result, the set of normalized weight vectors is generated in line 18.

For fast implementation, the Tabu list can be represented by a hash table [17], in which the key is the product of all prime numbers of a candidate solution module the size of the Tabu list (it must also be a prime number). Then, the store and seek operations can be done in $\mathcal{O}(1)$.[8]

The complexity of this algorithm is ruled by the generation of new solutions in $\mathcal{O}(Sm\log(m-1))$ and the evaluation of the measure in $\mathcal{O}(l^2m)$. However, $S \ll l$, thus the complexity is $\mathcal{O}(l^2m)$.

When the number of H_l is too small, the nearly uniform design obtained by this algorithm may be far from the uniform design. The cardinality of H_l can be determined by the Euler function $\phi(l)$.[9] For example, $\phi(n) = n-1$ if n is a prime, and $\phi(l) < l/2$ if l is even.

Uniform design overcomes the drawbacks that the random and the simplex-lattice design have, it is relatively easy to implement and its computational complexity is polynomial. An example of this approach is depicted in Figure 4.4-(d) for $m = 3$, $l = 21$, $T = 1000$ and $S = 20$.

[8] If there are collisions, the vectors can be concatenated as strings.

[9] Let $l = p_1^{r_1} \cdots p_t^{r_t}$ be the prime decomposition of l, where $p_1, ..., p_t$ are different primes and $r_1, ..., r_t$ are positive integers. Then $\phi(l) = l\left(1 - \frac{1}{p_1}\right) \cdots \left(1 - \frac{1}{p_t}\right)$

Algorithm 18 Main Loop of MOMBI

Input: MOP, termination condition, utility function u, set of weight vectors W
Output: Approximation set to the Pareto optimal front

1: $i \leftarrow 0$
2: Initialize population P_i
3: Evaluate population P_i
4: Calculate norms L_1 and L_2
5: Obtain reference points $\{\vec{z}^*, \vec{z}^{nad}\}$
6: Initialize $(\forall p \in P_i)$ $p.rank \leftarrow p.u^* \leftarrow \infty$
7: Execute $R2$ ranking algorithm $(P_i, \{u(\cdot : \vec{z}^*, w_j) | w_j \in W\})$
8: **while** termination condition is not fulfilled **do**
9:     Perform tournament selection
10:     Generate offspring P_i' using variation operators
11:     Evaluate population P_i'
12:     Calculate norms L_1 and L_2
13:     Update reference points $\{\vec{z}^*, \vec{z}^{nad}\}$
14:     Initialize $(\forall p \in P_i)$ $p.rank \leftarrow p.u^* \leftarrow \infty$
15:     Execute $R2$ ranking algorithm $(P_i \bigcup P_i', \{u(\cdot : \vec{z}^*, w_j) | w_j \in W\})$
16:     Reduce population $P_{i+1} \leftarrow \{P_i \bigcup P_i'\}$
17:     $i \leftarrow$i + 1
18: **return** P_i

4.4 Construction of a Metaheuristic

We are now ready to introduce the proposed approach in Algorithm 18, called MOMBI (Many-Objective Metaheuristic Based on the $R2$ Indicator). This approach is based on a Genetic Algorithm, and it first initializes the population by randomly selecting N solutions from the feasible space $\mathcal{F}$ using a uniform distribution.

In lines 3 to 5, we obtain the objective function values, the norms, and the reference points. In line 6 we initialize the variables rank and u^* for each individual to the worst values. In line 7, the ranking of the population is executed (see Algorithm 11). At each generation, the algorithm performs a binary tournament selection (see Algorithm 1), using the rank of each solution (line 9). In line 10, we use mutation and crossover operators to produce an offspring of N individuals. Again, in lines 11 to 13, we evaluate the objective functions, calculate the norms and the reference points are updated with the minimum and maximum objective function values. The structure of each individual is initialized in line 14. The parent and offspring population are ranked in line 15. In line 16, the reduction of the population is performed by selecting the best N candidates according to their rank, the best utility value obtained, and the norms.

It is worth indicating that this approach produces a finer-grained ranking (with fewer ties) than the nondominated sorting procedure adopted by NSGA-II.

The complexity of the tournament selection is $\mathcal{O}(N)$, as well as the offspring gen-

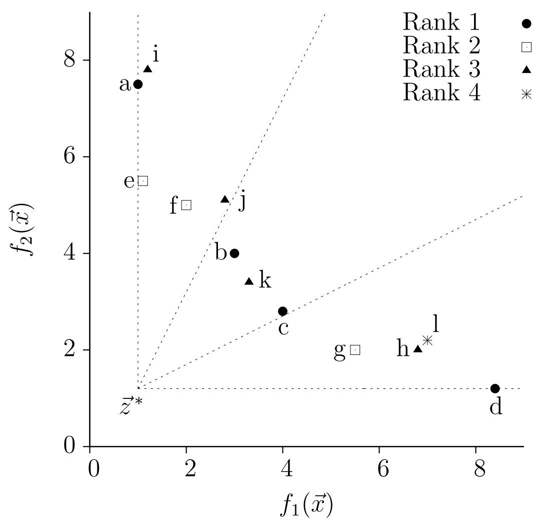


Figure 4.5: Illustration of the proposed ranking procedure based on the $R2$ indicator.

Table 4.1: A hypothetical example for MOMBI.

Solution	f_1	f_2	$u^*(\vec{a})$
a	1.0	7.5	0.00009
b	3.0	4.0	0.18018
c	4.0	2.8	0.16161
d	8.4	1.2	0.00010
e	1.1	5.5	0.01351
f	2.0	5.0	0.13513
g	5.5	2.0	0.12121
h	6.8	2.0	0.12121
i	1.2	7.8	0.02702
j	2.8	5.1	0.19696
k	3.3	3.4	0.20720
l	7.0	2.2	0.15151

eration. The evaluation of population, calculation of norms, upgrade of the reference points and the normalization can be performed in $\mathcal{O}(Nm)$ each. The initialization of the structure is made in $\mathcal{O}(N)$. As seen before, the ranking takes $\mathcal{O}(|W|N(\log N+m))$ and the reduction can be done in $\mathcal{O}(N\log|N|)$. Therefore, the overall complexity of MOMBI at each generation is $\mathcal{O}(|W|N(\log N+m))$. If the number of weight vectors $|W|$ is the same as the population size, the complexity becomes quadratic logarithmic for $m \ll N$, $\mathcal{O}(N^2(\log N+m))$. The storage is the same as the ranking algorithm, $\mathcal{O}(Nm)$.

In order to illustrate our proposal, we present here a hypothetical example of a bi-objective problem. We assume an approximation of the Pareto optimal set, which consists of twelve solutions, as shown in Figure 4.5. The dashed lines represent the weight vectors $\{(10^{-4},1),(1/3,2/3),(2/3,1/3),(1,10^{-4})\}$, the reference points are set to $\vec{z}^* = (1.0, 1.2)$ and $\vec{z}^{nad} = (8.4, 7.8)$. In Table 4.1, the objective functions and the optimum Tchebycheff value of each solution are shown. The first rank is formed with the solutions that are closest to the weights, according to the Tchebycheff metric, that is points $\{a, b, c, d\}$. The second rank consists of the remainder solutions, which are now closest to the weights, i.e., points $\{e, f, g\}$. The third rank consists of points $\{h, i, j, k\}$. Finally, the farthest solution l belongs to the last rank. It is worth noticing that, in this case, solutions g and h contribute equally to the weight $(10^{-4}, 1)$. However g has a lower Euclidean norm than h (5.8 vs 7.0) and is, therefore, considered to be better than h. If we wanted to select half of the solutions, using MOMBI, we would keep the individuals $\{a, b, c, d, e, g\}$, since e, f and g are in the same rank. In this case, we choose the solutions with the lowest Tchebycheff values. According to Pareto dominance, the nondominated solutions are $\{a, b, c, d, e, f, g, k\}$. Although f and k are nondominated, in this case $R2$ removes them with the aim of preserving diversity.

4.5 Integrating Preferences into MOMBI

When dealing with many-objective optimization problems it is impossible to cover the entire Pareto surface without increasing exponentially the population size. Therefore, the decision maker may be making poor resolutions as a result of the missing information. This weakness can be overcome by incorporating preferences into the MOEA. As we have seen in Chapter 1, this property is known as pertinency [41], and in the remainder of this section we will describe how to adapt MOMBI in order to guide the search only to the regions of main interest.

The proposed algorithm proceeds in two stages. In the first stage, the global Pareto optimal front is generated and then the decision maker is asked to provide the solutions that best suits his needs. In the second stage, these solutions are taken as reference points and for each of them, the population is ranked in such a way that the algorithm produces subregions of the Pareto optimal front. Unlike traditional interactive methods, the intervention of the decision maker comes after the maximum number of generations has elapsed, and not at each iteration. Figure 4.6 shows the flow chart of this proposal.

Let R be the set of reference points selected by the decision maker, and $\delta \in \mathbb{R}$ a positive scalar that tunes the density of the solutions around the reference points. In Algorithm 19, we present the modified version of MOMBI for handling preferences. In line 6 we establish the reference point in a lower position, subtracting the parameter δ given by the decision maker. The differences with respect to Algorithm 18 are in lines 8 to 9 and 17 to 18. Here, a solution gets a better rank if it is closer to a reference point. Thus, the individuals tend to be divided in subpopulations as shown in Figure 4.7.

The advantages of this method is that the decision maker does not need to specify ranges or hierarchies among the solutions; therefore it is intuitive to express user preferences. The complexity of the algorithm is $\mathcal{O}(|W||R|N(\log N + m))$, where N is the population size.

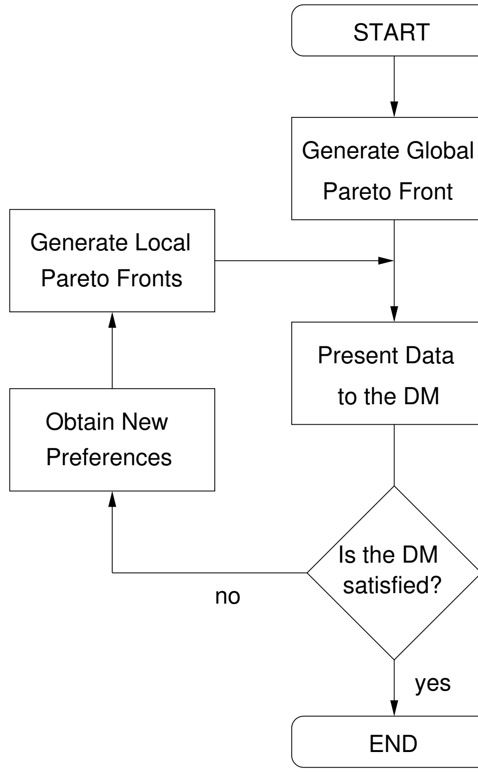


Figure 4.6: Flow chart of the proposed interactive method.

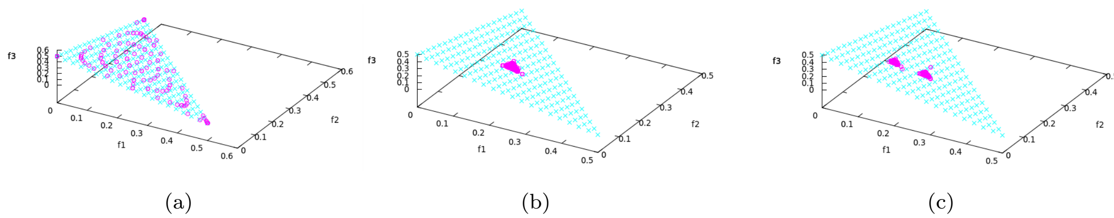


Figure 4.7: Decision maker's preference in the DTLZ1 problem: (a) normal execution, (b) at point $(0.16, 0.16, 0.16)$, (c) at points $(0.2, 0.15, 0.15)$ and $(0.12, 0.14, 0.24)$

4.6 Summary

In this chapter, we have introduced a new multi-objective evolutionary algorithm, named MOMBI, whose selection mechanism is based on the $R2$ indicator. It is worth emphasizing that the proposed approach is entirely based on the $R2$ indicator, since it does not incorporate Pareto dominance anywhere. MOMBI ranks individuals using utility functions. When there is a tie, the L_2 and L_1 norms are used as a secondary selection criterion.

The utility functions can be represented by the weighted L_p metrics and some variants that are based on optimization techniques. The main features of the proposed approach are summarized in Table 4.2.

Algorithm 19 Interactive MOMBI

Input: MOP, termination condition, utility function u, set of weight vectors W, set of reference points R, density of solutions δ

Output: Approximation set to the Pareto optimal front

$i \leftarrow 0$
Initialize population P_i
Evaluate population P_i
Calculate norms L_1 and L_2
Obtain reference points $\{\vec{z}^*, \vec{z}^{nad}\}$
Set $\vec{r} \leftarrow \vec{r} - \delta$
Initialize $(\forall p \in P_i)$ $p.rank \leftarrow p.u^* \leftarrow \infty$
for all $\vec{r} \in R$ **do**
    Execute $R2$ ranking algorithm $(P_i, \{u(\cdot : \vec{r}, w_j) | w_j \in W\})$
while termination condition is not fulfilled **do**
    Perform tournament selection
    Generate offspring P_i' using variation operators
    Evaluate population P_i'
    Calculate norms L_1 and L_2
    Update reference points $\{\vec{z}^*, \vec{z}^{nad}\}$
    Initialize $(\forall p \in P_i)$ $p.rank \leftarrow p.u^* \leftarrow \infty$
    for all $\vec{r} \in R$ **do**
        Execute $R2$ ranking algorithm $(P_i \bigcup P_i', \{u(\cdot : \vec{r}, w_j) | w_j \in W\})$
    Reduce population $P_{i+1} \leftarrow \{P_i \bigcup P_i'\}$
    $i \leftarrow$i + 1
return P_i

There are several methods for generating the weight vectors that come with the utility functions. The main features are also summarized in Table 4.3.

The runtime complexity of MOMBI is $\mathcal{O}(|W||R|N(\log N + m))$ and the required storage is $\mathcal{O}(Nm)$, where $|W|$ is the number of weight vectors, $|R|$ represents the number of reference points, N is the population size and m is the number of objectives.

Finally, the preferences of the decision maker can be incorporated intuitively into MOMBI, just providing a reduced set of meaningful solutions to the decision maker. Additionally, the user can specify the density around the reference points, thus allowing him to regulate the number of solutions that are generated within the region of his interest.

Table 4.2: Main features of the utility functions.

Metric	**Geometry**	**Generated Solutions**	**Extra Parameters**
L_1	convex and continuous	Pareto Optimal	no
L_2	convex and linear	Pareto Optimal	no
L_∞	Any	Weakly Pareto Optimal	no
Augmented	Any	Pareto Optimal	ρ, augmentation
Modified	Any	Pareto Optimal	ρ, augmentation
PBI	Any, more uniform	Pareto Optimal	θ, penalization

Table 4.3: Main features of the design of weight vectors. Here m represents the number of objectives.

Design	**Distribution**	**Implementation**	**Computational Complexity**	**Extra Parameters**
Randomized	none	easy	$\mathcal{O}(lm)$	l, weight vectors
Simplex-Lattice	poor	easy	$\mathcal{O}((h+m)^{m-1})$	h, proportion
Hypervolume-based	good, not uniform	medium	$\mathcal{O}(Tl^m)$	l, weight vectors T iterations $\vec{z}$ reference point
Uniform	good, nearly uniform	medium	$\mathcal{O}(l^2m)$	l, weight vectors T iterations S neighborhood size

Chapter 5

Experimental Study

In this chapter we investigate the efficacy of our proposed MOMBI with respect to that of seven state-of-the-art MOEAs and some of its variants, from two to ten objectives. Further, we examine the behavior of MOMBI when varying parameters specific to the $R2$ indicator, such as utility functions, design of the weight vectors, and normalization of the objective functions.

The experimental methodology is based on nonparametric statistics (see Appendix B for a quick introduction). In order to determine outperformance of optimizers, we employed one of the most popular, one-tailed test in the area of multi-objective optimization, the Wilcoxon rank sum test. Here, we rely on the R-project package to compute the test values.

All the experiments were executed 100 times and independently to each other, so that data is not reused in inferential analysis. The study is shown in Appendix C.[1]

The experiments were conducted on identical PCs having Intel(R) Core(TM) i7 processors running at 2.67GHz and with 3.8 GBytes in RAM, under Linux.

In Section 5.1, we established the sixteen test problems adopted for the comparative studies, as well as their features and parameters employed. In Section 5.2, we specify the performance indicators. In Section 5.3, we describe the conditions of the first experiment, in which we compare different state-of-the-art MOEAs with respect to MOMBI. In the analysis, we discuss the results of each optimizer. In Section 5.4, we investigate the behavior of MOMBI when varying utility functions. Here, we examine the performance of the utility functions described in Section 4.2, in each of the test instances. In Section 5.5, we analyze the sensitivity to the design of weight vectors, using the different approaches defined in Section 4.3. Finally, we conclude in Section 5.6 with some observed patterns of the experiments made.

[1] Due to its extent, the complete version is included on the CD that accompanies this report or is available for download at: `http://computacion.cs.cinvestav.mx/~rhernandez/mombi`.

Table 5.1: Properties of the test problems.

Problem	**Separability**	**Modality**	**Geometry**	**Bias**
DTLZ1	separable	multi	linear	no
DTLZ2	separable	uni	concave	no
DTLZ3	separable	multi	concave	no
DTLZ4	separable	uni	concave	polynomial
DTLZ5	unknown	uni	arc, degenerated	parameter dependent
DTLZ6	unknown	uni	arc, degenerated	parameter dependent
DTLZ7	$f_{1:m-1}$ not applicable f_m separable	$f_{1:m-1}$ uni f_m multi	disconnected, mixed	no
WFG1	separable	uni	$f_{1:m-1}$ convex f_m mixed	polynomial, flat
WFG2	non-separable	$f_{1:m-1}$ uni f_m multi	convex, disconnected	no
WFG3	non-separable	uni	linear, degenerated	no
WFG4	separable	multi	concave	no
WFG5	separable	deceptive	concave	no
WFG6	non-separable	uni	concave	no
WFG7	separable	uni	concave	parameter dependent
WFG8	non-separable	uni	concave	parameter dependent
WFG9	non-separable	multi, deceptive	concave	parameter dependent

Table 5.2: Configuration adopted for the WFG test suite.

Parameter	**Objective space dimension**								
	2D	3D	4D	5D	6D	7D	8D	9D	10D
position-related	4	4	6	8	10	12	14	16	18
decision variables	24	24	36	47	59	70	82	93	105

5.1 Test problems

For comparison purposes, we adopted the Deb-Thiele-Laumanns-Zitzler [210] and the Walking-Fish-Group [48] test suites (see Appendix A for their definition). All the minimization problems adopted are scalable with respect to the number of objectives and have a variety of geometries for the Pareto optimal front, such as linear, mixed (concave/convex), degenerated and disconnected. They also include some aspects such as separability and multi-frontality which make them more difficult to solve. In Table 5.1 we summarize the main features of these test problems [47].

In DTLZ, the total number of variables is given by $n = m + k - 1$, where m is the number of objectives. k was set to 5 for DTLZ1, 10 for DTLZ2-6 and 20 for DTLZ7. In WFG, the number of decision variables and the position-related parameter are shown in Table 5.2.

Table 5.3: Reference points for the test instances.

Test Problem	**Reference Point**
DTLZ1	$(1, 1, 1, \ldots)$
DTLZ2, DTLZ4	$(2, 2, 2, \ldots)$
DTLZ3	$(7, 7, 7, \ldots)$
DTLZ5	$(4, 4, 4, \ldots)$
DTLZ6	$(11, 11, 11, \ldots)$
DTLZ7	$(1, 1, 1, \ldots, 21)$
WFG	$(3, 5, 7, \ldots, 2m + 1)$

5.2 Performance Assessment

For comparing results, we selected the hypervolume indicator, which is equal to the sum of all the rectangular areas, bounded by some reference point. Since this reference point is important, we provide the values that we adopted for each test problem in Table 5.3. Mathematically, the hypervolume can be described using equation (5.1). It is worth noting that higher hypervolume values are preferred.

$$I_{HV}(A : z^{\vec{ref}}) = \left\{ \bigcup volume(v : z^{\vec{ref}}) | v \in A \right\}. \tag{5.1}$$

We used the algorithm proposed by Fonseca et al. in [175] for calculating the hypervolume from 2D to 9D,[2] and the one proposed by While et al. in [178], for 10 objectives, since it is a faster way of calculating exact hypervolumes.[3]

We also employed the $R2$ indicator in combination with the the Tchebycheff utility functions, where the reference point $\vec{z}^*$ is at the origin and the weight vectors were produced using the uniform design approach, described in Section 4.3.4 of Chapter 4. The number of weight vectors is proportional to the number of objectives, given by $|W| = m * 100$. As we defined before, the $R2$ indicator is given by equation (5.2). It is worth noting that lower values are preferred.

$$R2(A, W) = \frac{1}{|W|} \sum_{\mathbf{w} \in W} \min_{\mathbf{a} \in A} \left\{ \max_{i \in \{1, \ldots, m\}} w_i \left| a_i - z_i^* \right| \right\}. \tag{5.2}$$

Additionally, we also considered the running time for comparison purposes of MOEAs, measured in milliseconds. Running times are particularly relevant in this case, since we are interested in analyzing the way in which each of the algorithms behaves when increasing the number of objectives, and this includes measuring their computational cost.

[2]The source code is available at `http://iridia.ulb.ac.be/~manuel/hypervolume`.

[3]We used the 1.03 version, available at `http://www.wfg.csse.uwa.edu.au/hypervolume`.

5.3 Experiment 1: Comparison of MOEAs

In this experiment, we compared the performance of MOMBI with respect to that of seven state-of-the-art MOEAs, which have already been mentioned in Chapters 2 and 3. In the following, we provide a brief review and the details of the parameters settings:

1. **NSGA-II:** the Nondominating Sorting Genetic Algorithm II [13] ranks individuals using a nondomination criterion and a crowding distance for maintaining diversity (see Section 2.7.2 in page 31). We used version 1.1.6 for real-numbers encoding, available at `http://www.iitk.ac.in/kangal/codes.shtml`.

2. **MOEA/D:** the Multi-Objective Evolutionary Algorithm based on Decomposition [14] transforms an optimization problem into a number of scalar optimization subproblems that are simultaneously optimized (see Section 3.5.1 in page 54). The Nadir point is employed as a reference point, which is updated at each generation. The weight vectors are generated using the simplex-lattice design. We analyzed this algorithm using three utility functions: the weighted Tchebycheff metric (MOEA/D-TCH), normalized weighted Tchebycheff metric (MOEA/D-NTCH), and the PBI approach (MOEA/D-PBI). We used a neighborhood size of 20, and the implementation from 2007 for continuous search spaces, available at `http://dces.essex.ac.uk/staff/zhang/webofmoead.htm`.

3. **SMS-EMOA:** the S Metric Selection-Evolutionary Multi-objective Optimization Algorithm [3] is a popular hypervolume-based MOEA, that adopts nondominated sorting as its primary selection criterion and the hypervolume contribution as its secondary criterion (see Section 3.2.2 in page 44). Since SMS-EMOA requires a considerably large computational time in problems of high dimensionality [9], we used here a version that incorporates the algorithm proposed in [10] for estimating the hypervolume using Monte Carlo sampling, instead of the exact hypervolume calculations adopted in the original algorithm [3], for MOPs having 4 or more objectives. The number of samples was set to 10^5.

4. **Δ_p-DDE:** This MOEA, proposed in [9], uses the Δ_p indicator as the selection mechanism of differential evolution. The approximation to the true Pareto optimal front is made by means of the discretization of the nondominated solutions, using a resolution parameter that influences the quality of the outcome sets, as well as its computational cost (see Section 3.3.2 in page 46). We set this parameter to 120, 11, 5, 4 and 3 for 2D, 3D, 4D, 5D and 6-10D, respectively.

5. **$R2$-MOGA:** This genetic algorithm, proposed in [16], integrates the $R2$ indicator in a modified version of the nondominating sorting method of NSGA-II. It uses the weighted Tchebycheff metric and the utopian point as utility functions and reference point, respectively. The weight vectors change dynamically in each generation using the randomized design proposed by Jaszkiewicz

[197] (see Section 3.4.2 in page 50). We used the implementation from 2013, available at `http://www.tamps.cinvestav.mx/~adiazm`, and a modified version, called $R2$-MOGAw, in which the weights vectors are generated once, using the simplex-lattice design, with the purpose of studying the effects of dynamic weight generation.

6. **$R2$-MODE:** This approach is the same as $R2$-MOGA, but it uses differential evolution as its search engine (see Section 3.4.2 in page 50). We used the implementation from 2013, available at `http://www.tamps.cinvestav.mx/~adiazm`.

7. **$R2$-IBEA:** The $R2$ Indicator Based Evolutionary Algorithm [15] eliminates dominance ranking in its selection mechanism and performs indicator-based selection with the $R2$ indicator, in combination with the weighted Tchebycheff metric. The reference point is updated at each generation according to the extent of the population, and the vector generation method is based on the hypervolume (see Section 3.4.3 in page 51). Here, we used as weight vector generation the hypervolume design for 2 up to 7 objectives, and the simplex-lattice design for 8 up to 10 objectives (with reference point at $(2, 2)$. This was done because the hypervolume takes too much time to be computed for high dimensionality.

For MOMBI, the Nadir point is employed as the reference point, and we update it at each generation. The weight vectors are generated using the simplex-lattice design. We considered, the same utility functions as in MOEA/D: the weighted Tchebycheff metric (MOMBI-TCH), normalized weighted Tchebycheff metric (MOMBI-NTCH), and the PBI approach (MOMBI-PBI).

We performed 100 independent runs of each of the thirteenth MOEAs compared, in each of the test instances adopted. All the algorithms were implemented using real-numbers encoding in C/C++, except for $R2$-IBEA, which was implemented in Java 1.6.

The variation operators adopted for NSGA-II, MOEA/D, SMS-EMOA, $R2$-MOGA, $R2$-IBEA and MOMBI were: simulated binary crossover (SBX) and polynomial-based mutation [64] (see Section 2.7 in page 27). According to [13], the crossover rate was set to 0.9, while the mutation rate was set to $1/n$ (here, n represents the number of variables). The distribution indexes for both SBX and the polynomial-based mutation were set to 20. In the algorithms $R2$-MODE and Δ_p-DDE, based on Differential Evolution, the value for both F and CR was set to 0.5 [211, 212].

The population size and the maximum number of generations adopted in the experiment are shown in Table 5.4, and varied according to the value of m (i.e., number of objectives) and h (parameter of proportion in the simplex-lattice design, see Section 4.3.2 in page 66). The total number of function evaluations was set in such a way that it did not exceed 50,000. Following the proposal described in [14], in MOEA/D, $R2$-MOGA, $R2$-MODE, $R2$-IBEA and MOMBI, the number of weight vectors is the same as the population size.

Table 5.4: Parameters.

m	h	**Population Size**	**Generations**	**Function Evaluations**
2	119			
3	14	120	416	49920
4	7			
5	5	126	396	49896
6	4			
7		84	595	49980
8	3	120	416	49920
9		165	303	49995
10		220	227	49940

Discussion of Results

In Tables 5.1 and 5.2, we present the outperformance percentage of all the compared MOEAs in the DTLZ and the WFG test suites. These values were obtained counting the number of times in which one algorithm outperforms another one in the Wilcoxon Rank Sum Test with respect to the hypervolume indicator using a significance level of 5%. If no information is shown, it means that an algorithm did not outperform another one, or that the data is not available due to lack of convergence. The bars in each plot are sorted by algorithm. The last row and column summarize information by dimension and test problem, respectively.

In this case, the $R2$ indicator was omitted in the statistical test, since it revealed similar results to the hypervolume indicator, giving a slight advantage to those algorithms based on this indicator. Therefore, the information of the $R2$ indicator was used only to corroborate outliers. In the following paragraphs, we discuss our findings derived from this experiment.

As we expected, in DTLZ, the performance of NSGA-II decreases as the number of objectives increase. This algorithm completely misses convergence in the multi-modal problems DTLZ1 from 5 to 10 objectives and DTLZ3 from 4 to 10D. Even in low dimensionality (2D and 3D), NSGA-II is outperform by SMS-EMOA, $R2$-IBEA, MOEA/D-TCH, MOEA/D-PBI, MOMBI-TCH, and MOMBI-PBI. In DTLZ7, it gets acceptable performance for two, three, and apparently ten objectives. However, for the latest, the $R2$ indicator reveals no improvement. In the WFG test suite, NSGA-II obtains poor performance, surpassing only MOEA/D-NTCH. Only in WFG3, this algorithm obtains competitive results from 3 to 10 objectives, outperforming MOEA/D-PBI, $R2$-MODE, MOMBI-PBI, Δ_p-DDE, MOEA/D-TCH, MOMBI-TCH, and SMS-EMOA. In general, this algorithm gets the worst rank of all the compared MOEAs in the DTLZ test suite and ranks twelfth place in the WFG test problems.

In the DTLZ test problems, MOEA/D-TCH gets good results for 2 objectives. On average, this algorithm outperforms NSGA-II, MOEA/D-NTCH, SMS-EMOA, and Δ_p-DDE. In the WFG test suite, MOEA/D-TCH gets results that are not too im-

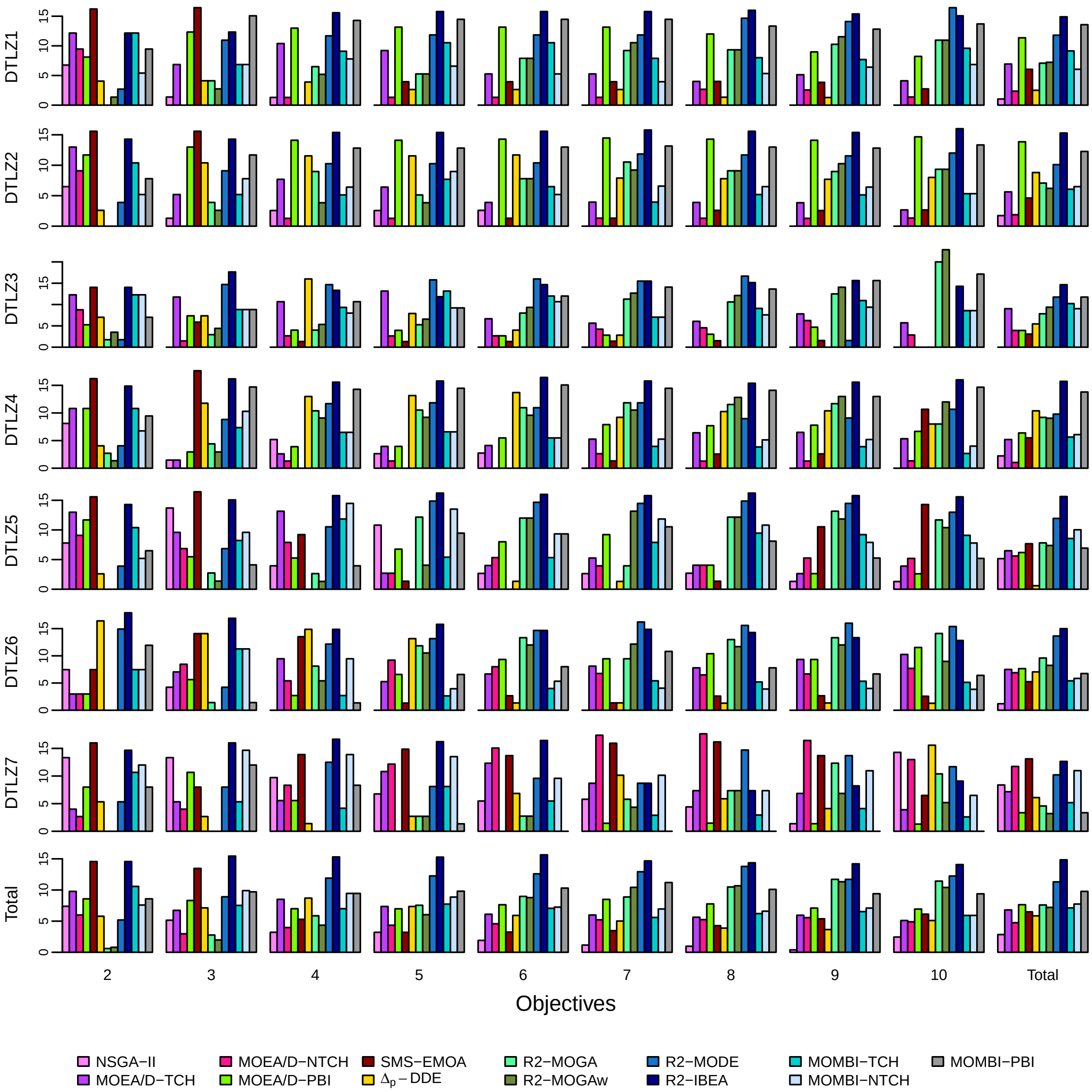


Figure 5.1: Outperformance percentage of MOEAs in the DTLZ test suite with respect to the hypervolume indicator using a significance level of 5% in the Wilcoxon Rank Sum Test.

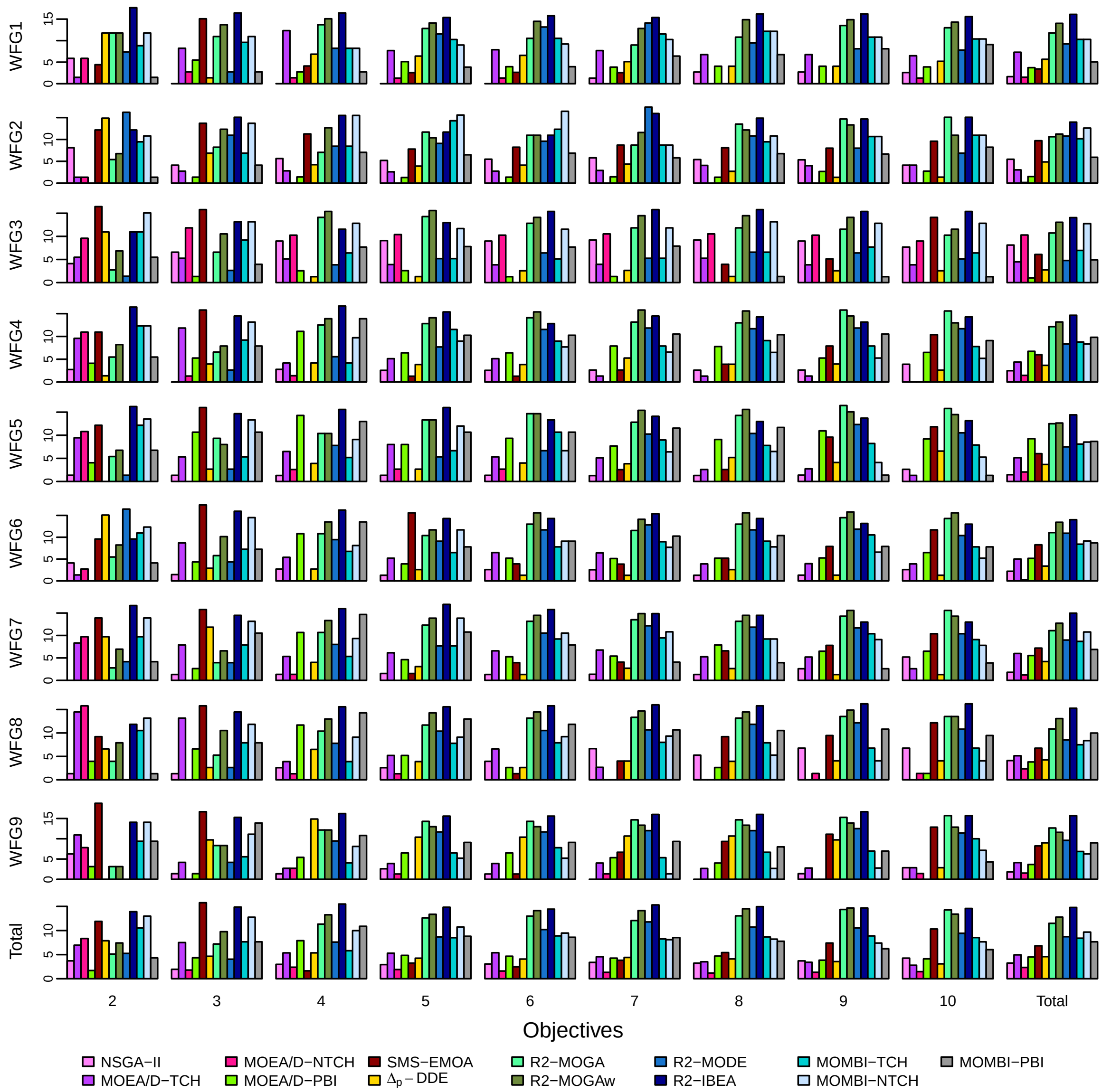


Figure 5.2: Outperformance percentage of MOEAs in the WFG test suite with respect to the hypervolume indicator using a significance level of 5% in the Wilcoxon Rank Sum Test.

pressive, outperforming MOEA/D-NTCH, NSGA-II, Δ_p-DDE, and MOEA/D-PBI. In more than five objectives, it loses diversity, since several of the objectives converge to zero values. In general, MOEA/D-TCH ranks ninth in both, the DTLZ and the WFG test suites.

MOEA/D-NTCH outperformed only NSGA-II, obtaining poor performance from 3 to 10 objectives in DTLZ1, DTLZ2, DTLZ3, DTLZ4 test problems. In DTLZ7 from 5 objectives onwards, it gets good results. In the WFG test suite, this algorithm obtains good results only in 2 objectives, outperforming NSGA-II, MOEA/D-TCH, MOEA/D-PBI, R2-MOGA, R2-MOGAw, R2-MODE, and MOMBI-PBI. In the WFG3 test problem, it obtains acceptable results from 2 to 10 objectives, being outperformed by $R2$-MOGA, $R2$-MOGAw, $R2$-IBEA, MOMBI-NTCH. From 3 objectives onwards, it loses diversity, since several of the objectives converge to zero values. In general, MOEA/D-NTCH ranks twelfth and thirteenth in the DTLZ and the WFG test suites, respectively.

MOEA/D-PBI outperforms the algorithms NSGA-II, MOEA/D-NTCH, SMS-EMOA, Δ_p-DDE, MOEA/D-TCH, and MOMBI-TCH in the DTLZ test suite. It gets good results in DTLZ1 and DTLZ2 test problems, and it faces convergence difficulties in DTLZ1 (9-10D), DTLZ3 (from 5-10D), DTLZ4 (wide variation in data), and DTLZ7 (from 5-10D). In WFG(4,5) from 3 to 10 objectives and in WFG(6-9) for 4D, MOEA/D-PBI gets acceptable performance, being outperformed by $R2$-MOGA, $R2$-MOGAw, $R2$-IBEA, and MOMBI-PBI. In general, MOEA/D-PBI ranks fifth in DTLZ and eleventh in the WFG test suites.

SMS-EMOA has a very good performance in low dimensionality and in almost all objectives of the DTLZ7 test problem. However, its performance goes down from 4 to 10 objectives, since it loses convergence, mainly in the multimodal problem DTLZ3, and in smaller proportion, in the DTLZ1, DTLZ5 and DTLZ6 test problems. In the DTLZ2 (4-10D) and DTLZ4 (4-9D) test problems, SMS-EMOA loses diversity, since the outcome sets do not cover the entire Pareto optimal front. In the degenerated DTLZ5 (9-10D) and biased DTLZ4 (10D) test problems, SMS-EMOA recovers performance, maybe because the population size increases. Moreover, in the DTLZ test suite, it gets the highest standard deviations, which means that its behavior is not very robust. In the WFG test suite, SMS-EMOA gets the best performance for 3 objectives, and competitive results for 2 and 10 objectives in WFG(3-9), since it loses convergence. In general, it is outperformed by $R2$-MOGA, $R2$-MOGAw, $R2$-MODE, $R2$-IBEA, MOMBI-TCH, and MOMBI-NTCH. Summarizing, this algorithm ranks tenth and eight in the DTLZ and the WFG test problems, respectively.

Δ_p-DDE gets good results in DTLZ2 and DTLZ4 for 3 up to 10 objectives, being outperformed by $R2$-IBEA and MOMBI-PBI in all dimensions. In DTLZ3, it provides competitive results for 2 up to 4 objectives. In DTLZ5, this MOEA has difficulties with the degenerated Pareto optimal front, getting a wide variation in its performance. Surprisingly, in DTLZ6, which is a test problem harder than DTLZ5, Δ_p performs much better for 2 up to 5 objectives. In DTLZ7, Δ_p-DDE has difficulties, obtaining high standard deviations. Even though the hypervolume indicator shows better results in DTLZ7 for 10 objectives, the $R2$ indicator does not reveal any improvement. In DTLZ1, DTLZ3, DTLZ5 and DTLZ7, Δ_p-DDE loses convergence from 7, 5, 2 and 8 objectives, respectively. Δ_p-DDE obtains competitive results in 2 dimensionality for WFG1-3, WFG6-7, and in WFG9 test problems from 3 to 9 objectives. In WFG1, it gets good spread of the Pareto optimal optimal front. In general, this algorithm outperforms NSGA-II and MOEA/D-NTCH, ranking eleventh in DTLZ and tenth in the WFG test suites.

$R2$-MOGA and $R2$-MOGAw have poor convergence and distribution in low dimensionality. However, as the number of objectives increases, their performance also improves, obtaining competitive results from 4 or 6 to 10 objectives, in both, the DTLZ and the WFG test suites. In almost all problems of high dimensionality in the DTLZ test suite, these MOEAs are outperformed by $R2$-IBEA, MOMBI-PBI, and $R2$-MODE. In DTLZ3, $R2$-MOGAw and $R2$-MOGA obtained the best results. DTLZ7 is a hard problem for these algorithms, since they tend to obtain poor results with a high variability. In the majority of the WFG test problems, these algorithms are outperformed only by $R2$-IBEA. In WFG1, they also obtained competitive results in low dimensionality. There is not too much difference in the performance of these algorithms in the DTLZ test problems. In the WFG test suite, $R2$-MOGAw outperforms $R2$-MODE in the majority of the test instances. In almost all problems $R2$-MOGAw slightly produces more diversity in the outcome sets. In general, $R2$-MOGA ranks sixth, while $R2$-MOGAw ranks seventh in the DTLZ test suite, and $R2$-MOGAw ranks second, while $R2$-MOGA ranks third in the WFG test suite.

$R2$-MODE has poor convergence and distribution in low dimensionality. However, this algorithm scales well as the number of objectives increases. In DTLZ5 and DTLZ6, $R2$-MODE obtains very good approximations. In the remainder problems of the DTLZ test suite, it obtains competitive results, being overcome only by $R2$-IBEA and sometimes by MOMBI-PBI or MOMBI-NTCH. In DTLZ3, this MOEA loses convergence and diversity for 9 and 10 objectives. In the WFG test problems, $R2$-MODE obtains competitive results from 5 to 10 objectives, being overcome by $R2$-IBEA, $R2$-MOGAw, $R2$-MOGA, and MOMBI-NTCH. In WFG6, for 2 objectives and WFG2 for 2 and 7 objectives, it achieves the best results. In WFG1, it gets good spread of the Pareto optimal optimal front. Only in WFG3, this algorithm has convergence difficulties. In general, $R2$-MODE ranks second in the DTLZ test suite and fifth in the WFG test suite.

$R2$-IBEA is the best optimizer, in both, the DTLZ and the WFG test suites. It obtains remarkable results in convergence, as well as in diversity. Even in low dimensionality, it is competitive with SMS-EMOA. It seems that in more than 7 objectives the simplex-lattice design for the weight vectors does not affect performance. It presents convergence difficulties in the DTLZ7 test problem, for 7 up to 10 objectives, where the standard deviations are high. Even though spread is poor in WFG1, it obtains the best convergence.

MOMBI-TCH produces approximation sets with good convergence and distribution in low dimensionality. In more than three objectives, it has low variation in data, outperforming NSGA-II, MOEA/D-NTCH, SMS-EMOA, Δ_p-DDE, and MOEA/D-TCH in several of the test instances of the DTLZ test suite. It presents difficulties of spread and convergence in DTLZ4, DTLZ6 and DTLZ7. In WFG, MOMBI-TCH gets competitive results, outperforming NSGA-II, MOEA/D-TCH, MOEA/D-NTCH, MOEA/D-PBI, Δ_p-DDE, SMS-EMOA, and MOMBI-PBI. Moreover, in high dimensionality, this algorithm tends to lose diversity, since some of the objectives converge to zero values. In general, MOMBI-TCH ranks eighth in the DTLZ test suite, and sixth in the WFG test suite.

MOMBI-NTCH has poor distribution of solutions in low dimensionality, due to normalization. In high dimensionality, it loses diversity, since several of the objectives converge to zero values. However, in DTLZ7, which is scaled in different units, this algorithm obtains competitive results from 2D to 7D. In the DTLZ test suite, this algorithm outperforms NSGA-II, MOEA/D-NTCH, SMS-EMOA, MOEA/D-TCH, Δ_p-DDE, MOEA/D-PBI, and MOMBI-TCH. This algorithm also outperforms MOMBI-PBI and $R2$-MODE in the WFG test suite, obtaining good performance in low dimensionality and in WFG2 and WFG3 for all objectives. In general, MOMBI-NTCH ranks fourth in the DTLZ test suite and the WFG test suite.

MOMBI-PBI produces uniform distributions of Pareto optimal fronts. In the DTLZ test suite, for 2 objectives, its results are competitive, and from 3D to 10D, it obtains good results, being outperformed only by $R2$-IBEA and $R2$-MODE. This algorithm faces convergence difficulties in DTLZ5, DTLZ6 and DTLZ7. In the WFG test problems, MOMBI-PBI outperforms MOEA/D-PBI, MOEA/D-NTCH, NSGA-II, Δ_p-DDE, and MOEA/D-TCH. Its best results are obtained in WFG4, WFG5, WFG6, and WFG8 from 4 to 8 objectives, and in WFG7 and WFG9 for 3 and 4 objectives. In high dimensionality it has convergence difficulties. In general, MOMBI-PBI ranks third in the DTLZ test suite and seventh in the WFG test suite.

With respect to the execution time, in Figure 5.3, we depict the average runtime in milliseconds of each optimizer. It is worth noting that these results agree with the computational complexity estimated in Chapter 3.

The computational cost increases very rapidly for SMS-EMOA in 2 and 3 objectives; this is because we are using the exact hypervolume calculation. Even though in more than 3 objectives, we are using an estimation method, the time consumption is the highest (only for 10 objectives, it lasts almost 3 hours to complete a single execution!). $R2$-IBEA is the second in terms of running time (1 hour and 20 minutes for 10 objectives). It is possible that the implementation made in Java worsens time. However, the slope increases from 7 objectives. On the other hand, Δ_p-DDE also increases a little bit the running time from 6 objectives (it takes 2.75 minutes for 10 objectives). Between $R2$-MOGA and $R2$-MODE there is not too much difference, they are stable, taking a low computational cost (on average, 12.2 seconds). Then, there is our proposed MOMBI, which increases in time similar to the last two algorithms (it requires 4.7s for 10 objectives). MOMBI's performance is followed by that of NSGA-II, and then MOEA/D, which, on average, take almost 2.2 seconds for 10 objectives. It is important to mention that, the inverted peak in 7 objectives is due

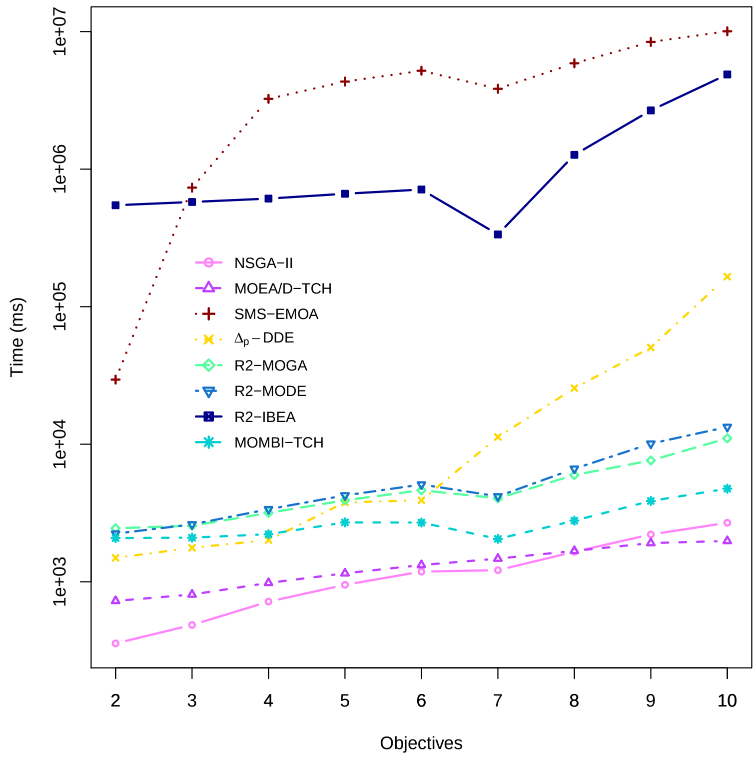


Figure 5.3: Average runtime of MOEAs.

to the reduction in population size (see Table 5.4).

5.4 Experiment 2: Performance of Utility Functions

In this experiment, we investigate the behavior of MOMBI when varying the utility functions. We consider the following approaches, which are based on the metrics described in Section 4.2, page 59:

- MOMBI-WS (weighted sum function)
- MOMBI-NWS (normalized weighted sum function)
- MOMBI-LS (metric of least squares)
- MOMBI-NLS (normalized metric of least squares)
- MOMBI-TCH (weighted Tchebycheff metric)
- MOMBI-NTCH (normalized weighted Tchebycheff metric)
- MOMBI-ATCH (augmented weighted Tchebycheff metric)

- MOMBI-NATCH (normalized augmented weighted Tchebycheff metric)
- MOMBI-MTCH (modified weighted Tchebycheff metric)
- MOMBI-NMTCH (normalized modified weighted Tchebycheff metric)
- MOMBI-PBI (Penalty-based Boundary Intersection metric)

In the case of MOMBI-ATCH, MOMBI-MTCH, and their normalized versions, the parameter ρ is set to 0.01. For MOMBI-PBI, the penalty parameter θ was set to 5. The remainder parameters settings are the same of those of the first experiment.

In Tables 5.4 and 5.5, we present the outperformance percentage of all the approaches in the DTLZ and the WFG test suites. These values were obtained counting the number of times in which one approach outperformed another one in the Wilcoxon Rank Sum Test with respect to the hypervolume indicator using a significance level of 5%. If no information is shown, it means that an algorithm did not outperform any other. The bars in each plot are sorted by approach. The last row and column summarize information by dimension and test problem, respectively.

As in the previous experiment, the $R2$ indicator was used only to corroborate hypervolume values. In the following, we discuss the observed results.

MOMBI-WS cannot deal with linear and concave Pareto optimal fronts, producing very poor diversity (only extreme values are found). In the DTLZ and the WFG test suites, it only outperforms MOMBI-NWS. It achieves its best results in WFG1 from 2 to 7 objectives, since it generates good spread, but the Pareto optimal front is biased. Moreover, in DTLZ7 for 5 up to 10 objectives, it obtains acceptable results. MOMBI-WS ranks tenth and eleventh in the DTLZ and the WFG test problems, respectively.

MOMBI-NWS obtains the best results in WFG1. It also obtains competitive results in WFG2 for 3, 4 and 5 objectives and in DTLZ7 from 6 to 10 objectives. In the same way as in MOMBI-WS, it cannot deal with linear and concave Pareto optimal fronts. MOMBI-NWS ranks eleventh in the DTLZ test suite and tenth in the WFG test suite.

MOMBI-LS obtains good results from 6 to 10 objectives in the DTLZ and the WFG test problems, being outperformed by MOMBI-TCH, MOMBI-NTCH and MOMBI-PBI. It gets competitive results in WFG1 and it can deal with linear Pareto optimal fronts. However, it faces difficulties of diversity in concave geometries. MOMBI-LS ranks sixth and fourth in the DTLZ and the WFG test problems, respectively.

MOMBI-NLS outperforms MOMBI-WS, MOMBI-NWS and MOMBI-NATCH in the DTLZ test problems, and outperforms MOMBI-NATCH in the WFG test suite. It can deal with linear Pareto optimal fronts, but it faces difficulties in concave geometries, and it produces bad distributions in low dimensionality. This approach obtains the best results in DTLZ7 from 4 objectives, and good performance in WFG1 and WFG3 from 2 objectives. MOMBI-NLS ranks ninth, in both, the DTLZ and the WFG test problems.

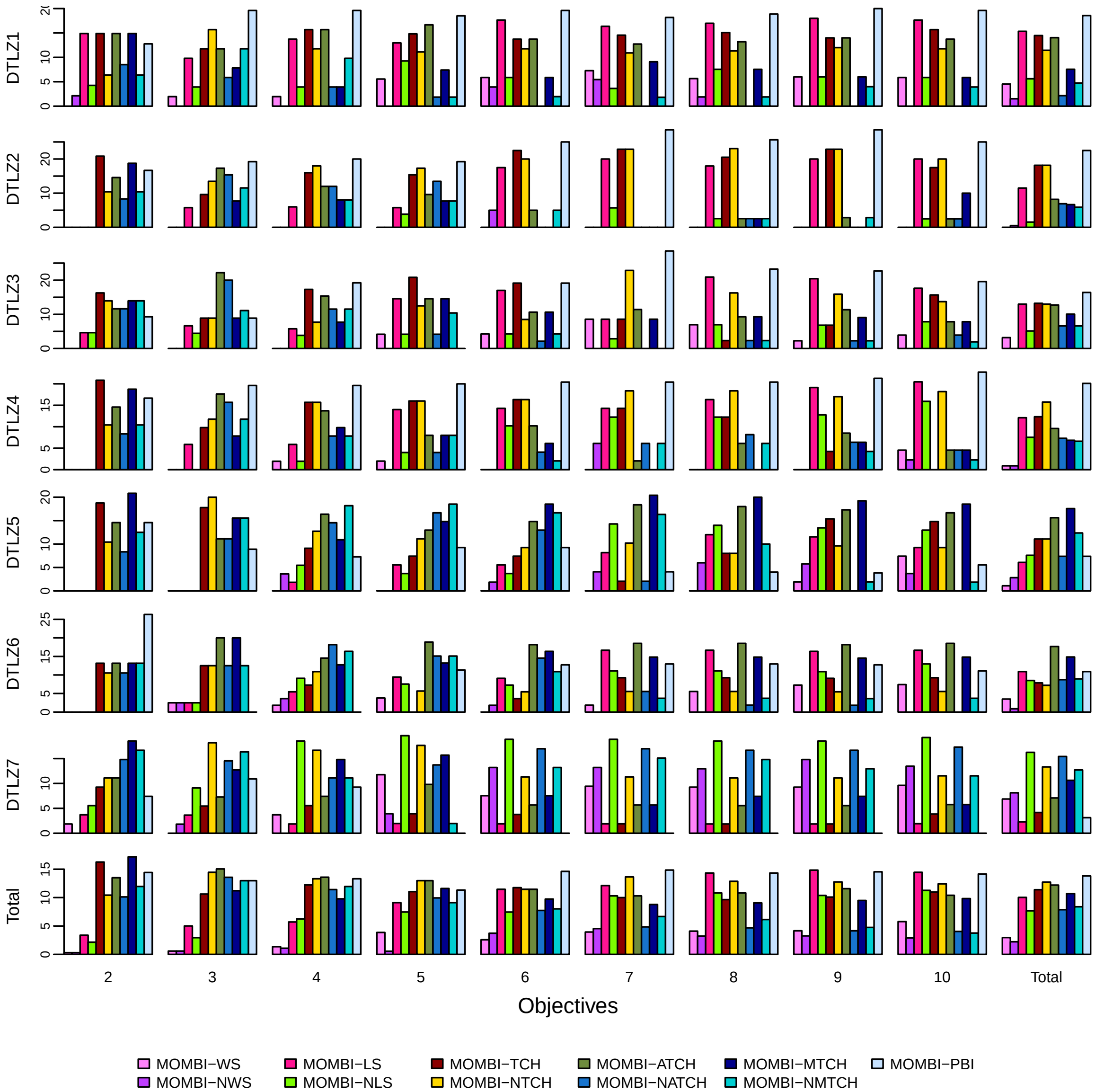


Figure 5.4: Outperformance percentage of utility functions in the DTLZ test suite with respect to the hypervolume indicator using a significance level of 5% in the Wilcoxon Rank Sum Test.

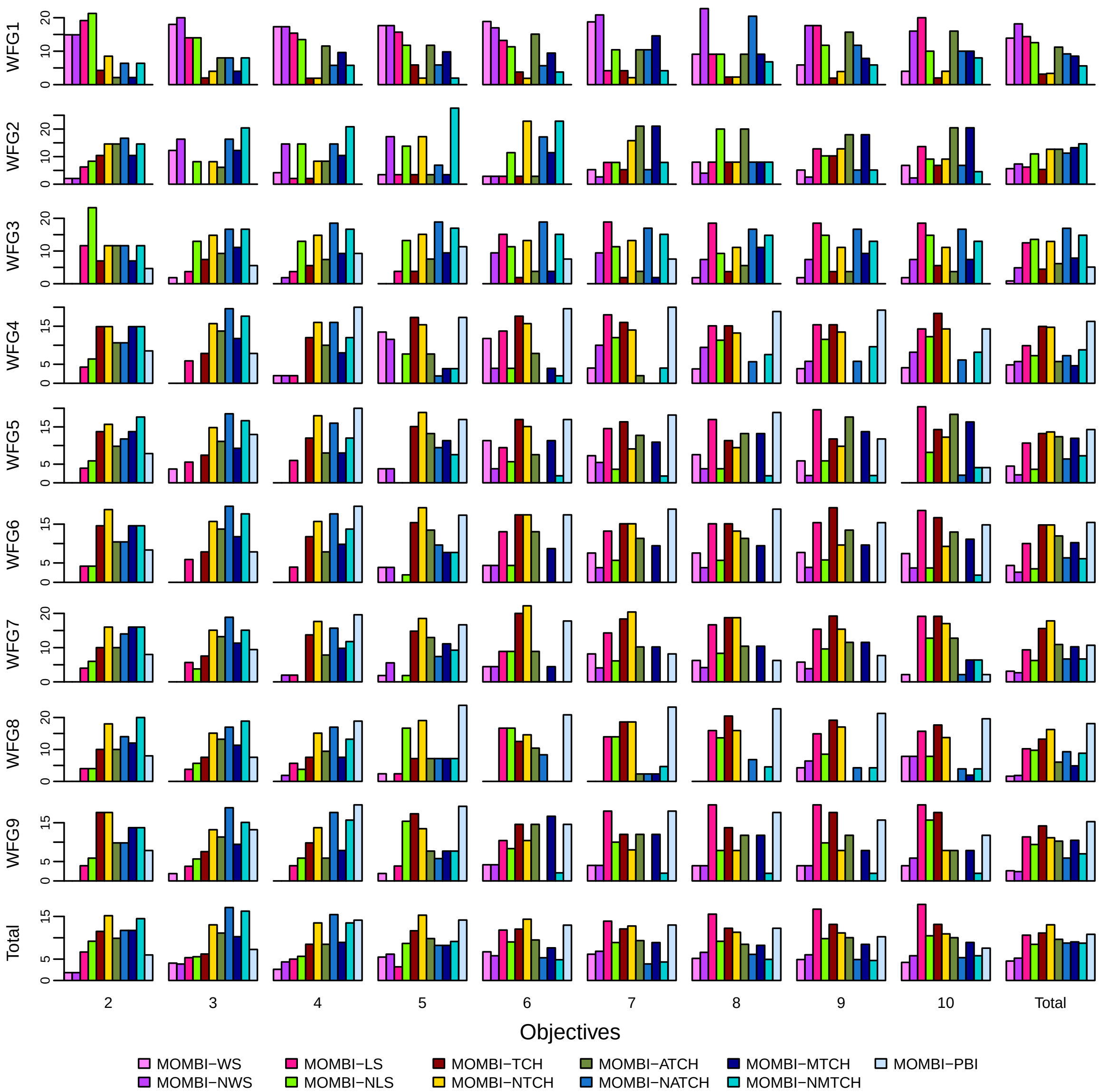


Figure 5.5: Outperformance percentage of utility functions in the WFG test suite with respect to the hypervolume indicator using a significance level of 5% in the Wilcoxon Rank Sum Test.

MOMBI-TCH is outperformed by MOMBI-PBI, MOMBI-NTCH and MOMBI-ATCH in the DTLZ test problems. In the WFG test problems, it is outperformed by MOMBI-NTCH. The main features of this approach have been described in the previous experiment. MOMBI-TCH ranks fourth in the DTLZ test suite and second in the WFG test suite.

MOMBI-NTCH ranks second and first in the DTLZ and the WFG test problems, respectively. It is outperformed by MOMBI-PBI and MOMBI-ATCH in the DTLZ test problems. In the WFG test suite it obtains the best results (for more features of this approach, the reader can review the previous experiment).

MOMBI-ATCH is outperformed only by MOMBI-PBI in the DTLZ test problems, and in the WFG test problems, it is also outperformed by MOMBI-LS, MOMBI-TCH and MOMBI-NTCH. In DTLZ6, it obtains the best results of convergence, it outperforms MOMBI-TCH and MOMBI-PBI. However, in general, the produced distributions are not uniform, since they are biased to the knee (see Definition 2.5.14) and edges, and in more than five objectives, it loses diversity, since several of the objectives converge to zero values. MOMBI-ATCH ranks third in the DTLZ test suite and fifth in the WFG test problems.

MOMBI-NATCH outperforms MOMBI-WS and MOMBI-NWS, in both, the DTLZ and the the WFG test problems. It obtains competitive results in the DTLZ test problems, WFG4-9 for 3 and 4 objectives. In WFG3, this approach gets the best results. In the same way as in MOMBI-ATCH, the distributions are not uniform. In more than five objectives, it loses diversity, since several of the objectives converge to zero values. MOMBI-NATCH ranks eighth in the DTLZ test suite and seventh in the WFG test suite.

MOMBI-MTCH is outperformed by MOMBI-TCH, MOMBI-NTCH, and MOMBI-ATCH in the DTLZ test suite. In the WFG test problems, this approach outperforms MOMBI-WS, MOMBI-NWS, MOMBI-NLS and MOMBI-NATCH. It obtains the best results in DTLZ5, outperforming even MOMBI-PBI, MOMBI-TCH and MOMBI-NTCH. Moreover, it gets competitive results in two objectives for all instances of the DTLZ test suite. In more than five objectives, this approach loses diversity, since several of the objectives converge to zero values. MOMBI-MTCH ranks fifth and sixth in the DTLZ and the WFG test problems, respectively.

MOMBI-NMTCH outperforms MOMBI-WS, MOMBI-NWS and MOMBI-NATCH in the DTLZ test suite, and also MOMBI-MTCH in the WFG test problems. This approach obtains competitive results for 3 up to 4 objectives, in both test problems. In low dimensionality, distributions are poor, and in more than five objectives, it loses diversity, since several of the objectives converge to zero values. MOMBI-NMTCH ranks seventh and eight in the DTLZ and the WFG test problems, respectively.

MOMBI-PBI ranks first in the DTLZ test suite and third in the WFG test problems. In the WFG test problem, this approach is outperformed by MOMBI-TCH and MOMBI-NTCH. The main features of this approach have been described in the previous experiment.

5.5 Experiment 3: Sensitivity to the Design of Weight Vectors

In this experiment, we analyze the sensitivity to the design of weight vectors in MOMBI, combined with the PBI approach as utility functions. Here, we consider the following designs, which were described in Section 4.3, page 64:

- MOMBI-RND (random design of Jaszkiewicz [197])
- MOMBI-SLD (simplex-lattice design)
- MOMBI-HV (hypervolume based design)
- MOMBI-UD (uniform design)

In all cases, the weight vectors were generated once. In MOMBI-HV, we only consider up to 7 objectives, due to the high time consumption in the weight vector generation. The number of iterations was set to 5000 (see Algorithm 15). In MOMBI-UD, the number of generations was set to 2000 (see Algorithm 16). It is worth noting that, MOMBI-SLD is equivalent to MOMBI-PBI. The remainder parameters settings are the same of those of the first experiment.

In Tables 5.6 and 5.7, we present the outperformance percentage of all the approaches in the DTLZ and the WFG test problems. These values were obtained counting the number of times in which one approach outperformed another one in the Wilcoxon Rank Sum Test with respect to the hypervolume indicator using a significance level of 5%. If no information is shown, it means that an algorithm did not outperformed any other, or that data is not available due to missing convergence. The bars in each plot are sorted by approach. The last row and column summarize information by dimension and test problem, respectively.

As in the first experiment, the $R2$ indicator was used to corroborate hypervolume values. We also used the algorithm proposed by While et al. in [178] for calculating the hypervolume indicator for 9 and 10 objectives (see Section 5.2 in page 81). In the following, we discuss the results.

MOMBI-RND presents poor diversity with a high variability. In the WFG test suite, it outperforms MOMBI-UD in the majority of the test instances and sometimes MOMBI-SLD. In WFG1, it obtains good performance from 5 to 10 objectives. This approach gets the worst rank in the DTLZ test suite, and ranks third in the WFG test suite.

MOMBI-SLD ranks first, in both, the DTLZ and the WFG test suites. In the DTLZ test problems, this approach obtained indicator values very close to those produced by MOMBI-HV, being the latter slightly better. It produces uniform distributions, but it loses convergence in more than 5 objectives. In 9 and 10 objectives of the WFG test suite, MOMBI-SLD is outperformed by MOMBI-RND.

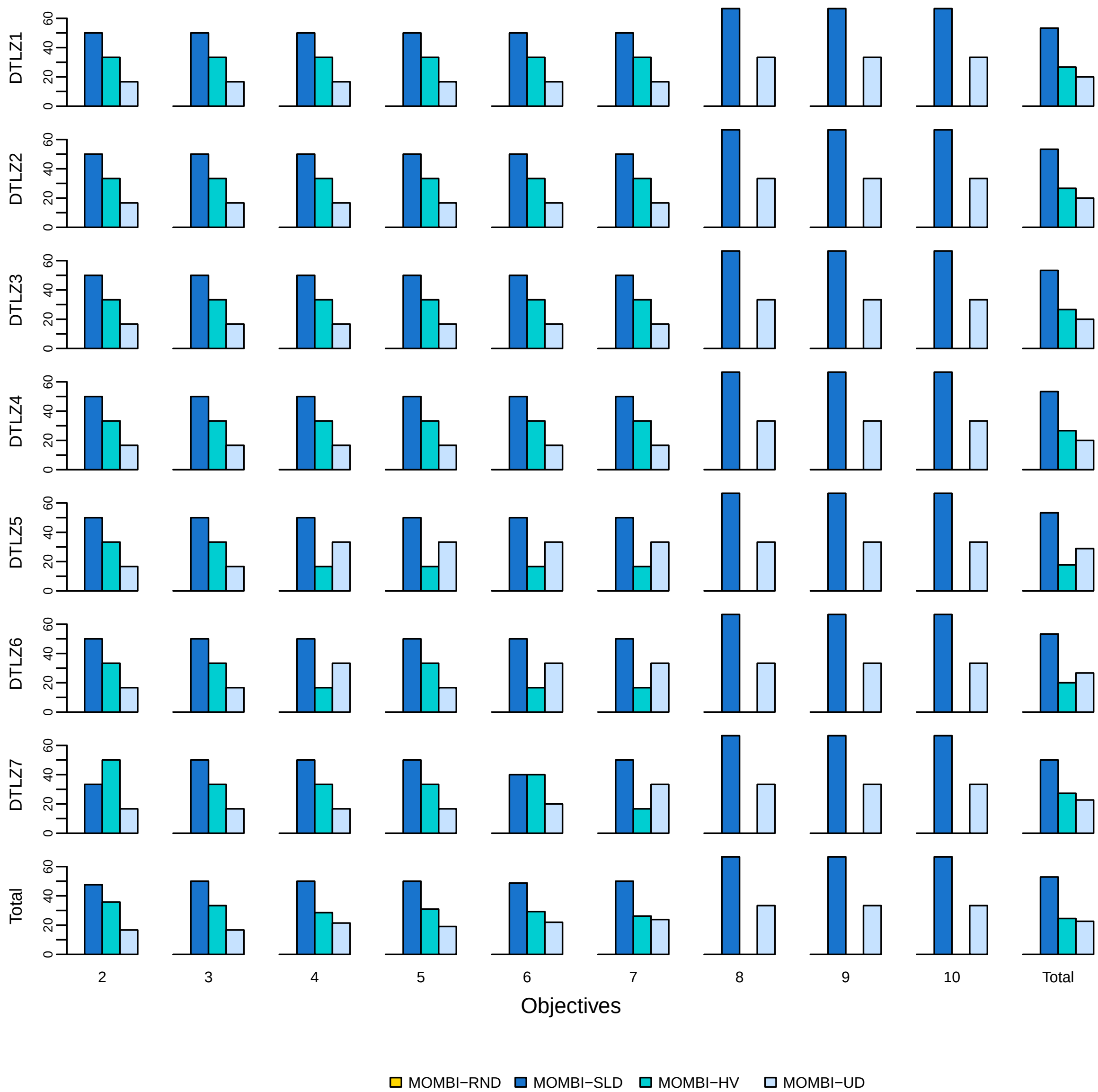


Figure 5.6: Outperformance percentage of weight vectors in the DTLZ test suite with respect to the hypervolume indicator using a significance level of 5% in the Wilcoxon Rank Sum Test.

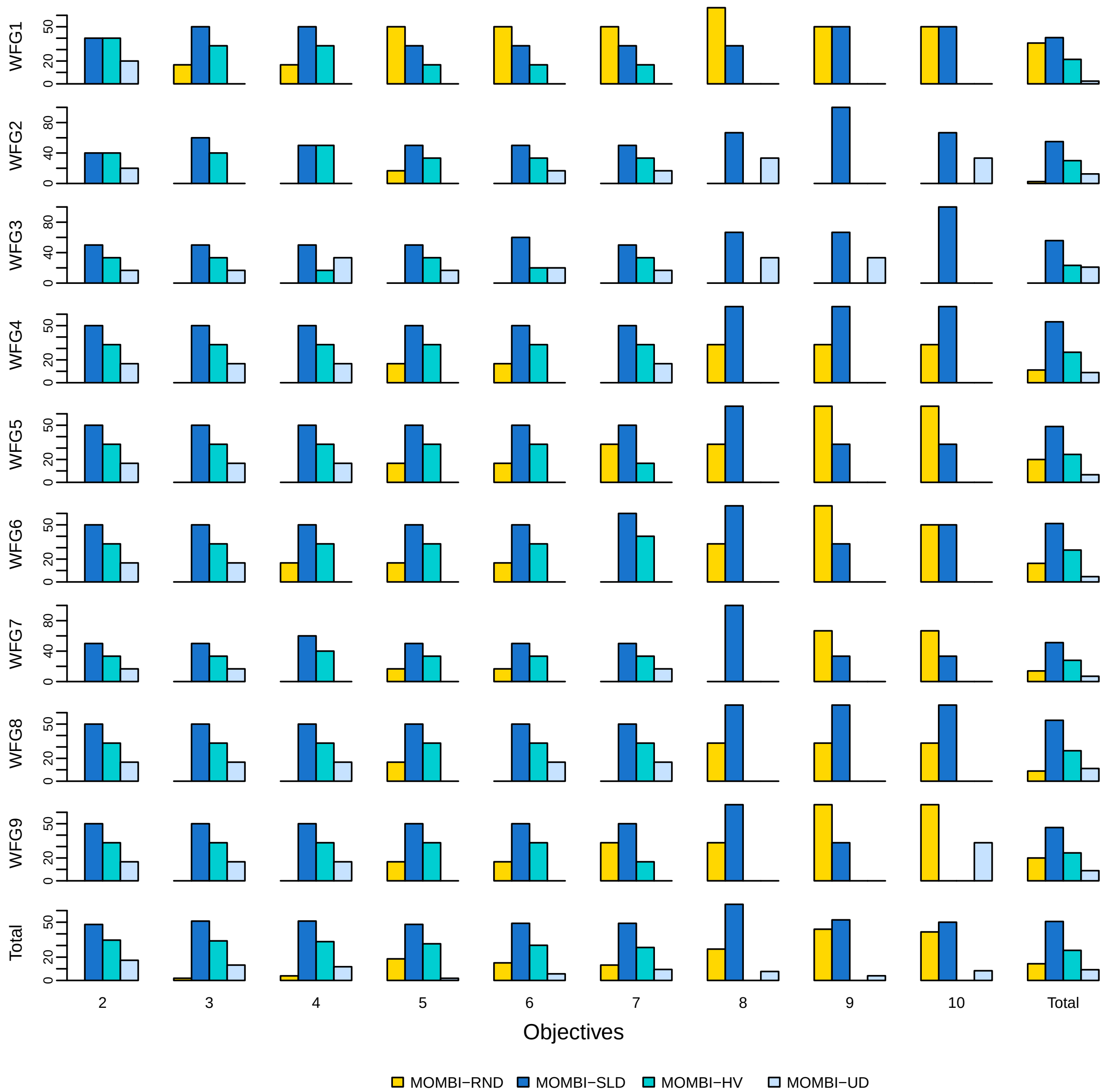


Figure 5.7: Outperformance of weight vectors in the WFG test suite with respect to the hypervolume indicator using a significance level of 5% in the Wilcoxon Rank Sum Test.

MOMBI-HV outperforms MOMBI-SLD, ranking second, in both, the DTLZ and the WFG test problems. It produces good diversity, but as the number of objectives increases it loses uniformity, spread and convergence.

MOMBI-UD ranks third in the DTLZ test problems and fourth in the WFG test suite. It produces good distributions, but it loses spread from 3 objectives, since points at the boundary are missing. In DTLZ5 and DTLZ6, this approach loses diversity, as well as convergence from 5 and 3 objectives, respectively.

5.6 Summary

We present, in this chapter, the numerical results of the comparison made among the proposed MOMBI and some other state-of-the-art MOEAs.

These results indicate that MOMBI is a suitable optimizer for solving MOPs, requiring a low computational cost, and outperforming, in the majority of the instances of the DTLZ and the WFG test suites, to NSGA-II and Δ_p-DDE for any number of objectives; to MOEA/D for more than 2 objectives; and to SMS-EMOA for more than 3 objectives. MOMBI also outperforms $R2$-MOGA and $R2$-MODE in low dimensionality.

As we expected, NSGA-II performed poorly as the number of objective functions increased, while maintaining a low computational cost.

Δ_p-DDE was competitive with the compared algorithms and obtained good diversity in all dimensions. However, it faced convergence difficulties in multimodal, degenerated, deceptive and disconnected problems. We hypothesize that in the degenerated DTLZ6 test problem, this MOEA obtained good results due to the use of differential evolution, as it happened for $R2$-MODE. The computational cost of Δ_p-DDE is also low.

MOEA/D showed a deterioration in the diversity of its outcome sets as the number of objective functions increased, since several of these objectives tend to converge to zero values.[4] Diversity varies with respect to the utility functions used. The Tchebycheff approach produces poor distributions, and its normalized version worsens the spread of solutions. However, it seems that normalization improves performance in disconnected and degenerated problems. The PBI approach generates a uniform distribution of solutions, but it loses convergence in high dimensionality. All these issues are also present in MOMBI, but in a lower extent than in MOEA/D. On the other hand, MOEA/D obtained the lowest computational cost of all the approaches compared in our study.

SMS-EMOA showed a deterioration in the quality of its solutions as the number of objectives increased. Here, it is important to mention that the approximation method used for the hypervolume contributions produces acceptable solutions when the number of generations, population size and/or samples increases with respect to the number of objectives. However, its high computational cost makes it prohibitive.

[4] This observation was recently pointed out by Ishibuchi in [213].

On average, SMS-EMOA obtained the highest computational cost.

$R2$-MOGA and $R2$-MODE showed an improvement in their performance as the number of objective functions increased. In the majority of the test instances adopted, the random weight vector generation did not affect convergence, but it impacted diversity. It seems that the search engine has a significant impact on performance in some MOPs. $R2$-MOGA had difficulties with disconnected problems, while $R2$-MODE had problems with degenerated problems. The computational cost of these approaches is low.

It is remarkable to note that $R2$-IBEA was the best algorithm in the experiment. This approach produced good convergence, as well as diversity in its outcome sets, although this does not necessarily mean that such solutions were uniformly distributed along the Pareto optimal front. The weight vector generation based on hypervolume seems to be suitable up to 5 objectives. From 6 objectives onwards, diversity is degraded. Although the weight vectors are generated once, the computational cost involved in their generation is very high. An alternative choice could be to use an approximation method instead. The computational cost of $R2$-IBEA is still high, but this may be due to its implementation.

In this chapter, we also investigated the sensitivity of MOMBI when varying parameters specific to the $R2$ indicator.

The utility functions that performed best were PBI and normalized Tchebycheff metrics for the DTLZ and the WFG test problems, respectively. It is corroborated that the different choice of utilities helps to improve or worsen performance in some MOPs. As it occurred in MOEA/D, normalization also affects the performance of disconnected and degenerated problems.

Finally, the weight vector generation influences the performance of MOMBI. The approach that maximized the hypervolume indicator was the simplex-lattice design. In uniform design, the number of weight vectors is not restricted to a combinatorial number (as in simplex-lattice design), which makes it scalable with respect to the number of objectives. However, the quality of the Pareto optimal solutions produced did not maximize the hypervolume, because boundary points are not considered in this approach. An improvement of this design would include such points.

Chapter 6

Conclusion and Future Work

In this work, we have introduced a new multi-objective evolutionary algorithm whose selection mechanism is based on the $R2$ indicator. It is worth emphasizing that the proposed approach is entirely based on the $R2$ indicator, since it does not incorporate Pareto dominance anywhere. Therefore, the nondominated solutions must be filtered after the outcome sets are generated.

Our experimental results show that the proposed MOMBI is able to deal with hard problems, outperforming NSGA-II, MOEA/D, Δ_p-DDE, and the modified version of SMS-EMOA in most cases and that it requires a considerably lower computational cost than SMS-EMOA.

MOMBI also outperforms $R2$-MOGA and $R2$-MODE in low dimensionality. Even though, these approaches are similar, they produce different results. This, perhaps, is due to the fact that MOMBI introduces a high selection pressure. In contrast, $R2$-MOGA and $R2$-MODE are less elitist.

Diversity varies with respect to the utility functions used. Here, we noticed that the Tchebycheff approach produces poor distributions, and its normalized version worsens the spread of solutions. However, it seems that normalization improves performance in disconnected and degenerated problems. On the other hand, the PBI approach generates uniform distributions of solutions, but it loses convergence in high dimensionality.

MOMBI is also sensitive to the choice of the weight vector generation. Here, the simplex-lattice design obtained the best results. The use of uniform design gave acceptable results. but such results can be improved by adding boundary points.

The use of search engines, such as differential evolution, can influence the performance of some MOEAs.

Evidently, much more work is required. We are interested, for example, in incorporating a mechanism to handle constraints, but first we need to design restricted test problems, since the only known scalable problems are DTLZ8 and DTLZ9 [214], and none of them is deceptive, nonseparable or scaled in different units. Here, it would be interesting to modify the approach for generating the WFG test problems [48].

We are also interested in studying the sensitivity of our proposed approach to the choice of the reference set, in order to determine which approach performs better:

static, dynamic, ideal, utopian, and proportional, as in $R2$-IBEA. Additionally, we want to investigate the behavior of MOMBI when varying the number of weight vectors and the population size. We also would like to perform more experiments, using real-world problems and different statistical tests.

Finally, it would be interesting to combine this indicator with another one (e.g., Δ_p), with the aim of combining their advantages and compensate for their possible limitations.

Appendix A

Test Problems

With the aim of having a better understanding of the working principles of optimizers, there are several benchmarks that have been suggested in the field of multi-objective optimization [47]. These artificial test problems examine the ability to control difficulties in both converging to the true Pareto optimal front and in maintaining a widely distributed set of solutions. Moreover, they offer many advantages over real-world problems, such as scalability, knowledge of the exact shape and location of the resulting Pareto optimal front, fast execution time, as well as ease of understanding, implementation and visualization.

In this Appendix, we review two important test suites, which are commonly used in many-objective optimization, and are defined for real-valued and unconstrained problems.

The implementation in C++ and Java of the test problems described here, can be found in the following web pages:

PISA: `http://www.tik.ee.ethz.ch/pisa` (C++)

jMetal: `http://jmetal.sourceforge.net` (Java)

MOEA Framework: `http://www.moeaframework.org` (Java)

Walking Fish Group: `www.wfg.csse.uwa.edu.au/toolkit` (C++)

Shark machine learning library (DTLZ): `http://image.diku.dk/shark` (C++)

For downloading samples of the Pareto optimal front in 3D, the reader should consult the following web site[1]:

EMOOBOOK: `http://www.cs.cinvestav.mx/~emoobook`

Additional information, such as hypervolume values and formulation of the Pareto optimal front can be consulted in the site:

ETH Systems Optimization: `http://people.ee.ethz.ch/~sop/download/supplementary/testproblems`

[1] jMetal also provides this feature for 2D.

A.1 Deb-Thiele-Laumanns-Zitzler Test Suite

The Deb-Thiele-Laumanns-Zitzler (DTLZ) test suite [214, 83, 210] includes nine representative test problems for comparing optimizers, which are scalable to any number of decision variables and objectives. The majority of these problems are separable, including degenerated and multimodal Pareto optimal fronts, of which the exact shape and location are known.

In the following, we present the seven unconstrained problems of the DTLZ test suite. Here, the total number of decision variables is given by $n = m + k - 1$, where m represents the number of objectives and k is the number of distance parameters.

DTLZ1

This test problem is separable and multimodal. Its Pareto optimal front is linear and is given by the following expression:

$$\begin{aligned}
&\text{Given} && \vec{x} = \{x_1, \ldots, x_{m-1}, x_m, \ldots, x_n\} \\
&\text{Minimize} && f_1(\vec{x}) = 0.5\,(1 + g(\vec{y})) \prod_{i=1}^{m-1} x_i \\
& && f_{j=2:m-1}(\vec{x}) = 0.5\,(1 + g(\vec{y}))\,(1 - x_{m-j+1}) \prod_{i=1}^{m-j} x_i \\
& && f_m(\vec{x}) = 0.5\,(1 + g(\vec{y}))\,(1 - x_1) \\
&\text{where} && y_{i=1:k} = \{x_m, x_{m+1}, \ldots, x_n\} \\
& && g(\vec{y}) = 100\left[k + \sum_{i=1}^{k} (y_i - 0.5)^2 - \cos(20\pi(y_i - 0.5))\right] \\
&\text{subject to} && \forall i \in \{1, \ldots, n\} \quad 0 \leq x_i \leq 1.
\end{aligned} \tag{A.1}$$

All objective function values lie on the linear hyper-plane $\sum_{i=1}^{m} f_i = 0.5$. The Pareto optimal solution corresponds to $\vec{y} = (0, 0, \ldots)^T$, and a value of $k = 5$ is suggested here. The difficulty in this problem is to converge to the hyper-plane. The search space contains $(11^k - 1)$ local Pareto optimal fronts, each of which can attract an optimizer. The Pareto optimal front is shown in Figure A.1.

DTLZ2

This problem is separable and unimodal. The geometry of its Pareto optimal front is concave (see Figure A.2), and is defined as follows:

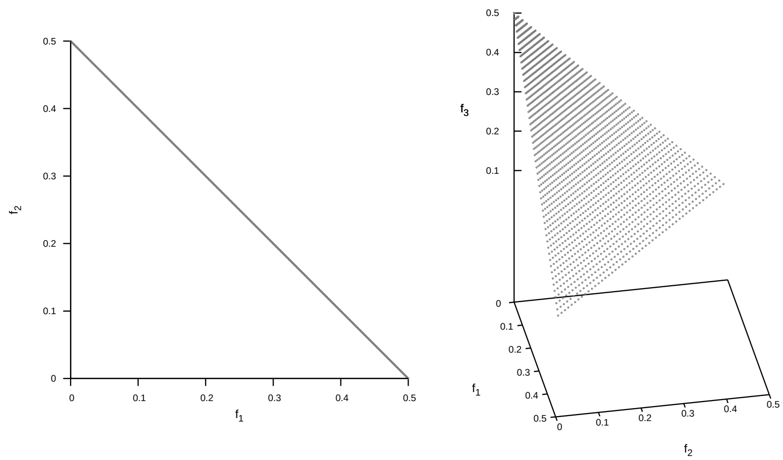


Figure A.1: DTLZ1 Pareto optimal front for two (left) and three (right) objectives.

$$
\begin{aligned}
\text{Given} \quad & \vec{x} = \{x_1, \ldots, x_{m-1}, x_m, \ldots, x_n\} \\
\text{Minimize} \quad & f_1(\vec{x}) = (1 + g(\vec{y})) \prod_{i=1}^{m-1} \cos(x_i \pi/2) \\
& f_{j=2:m-1}(\vec{x}) = (1 + g(\vec{y})) \left(\prod_{i=1}^{m-j} \cos(x_i \pi/2) \right) \sin(x_{m-j+1} \pi/2) \\
& f_m(\vec{x}) = (1 + g(\vec{y})) \sin(x_1 \pi/2) \\
\text{where} \quad & y_{i=1:k} = \{x_m, x_{m+1}, \ldots, x_n\} \\
& g(\vec{y}) = \sum_{i=1}^{k} (y_i - 0.5)^2 \\
\text{subject to} \quad & \forall i \in \{1, \ldots, n\} \quad 0 \leq x_i \leq 1.
\end{aligned}
\tag{A.2}
$$

The Pareto optimal solutions corresponds to $\vec{y} = (0.5, 0.5, \ldots)^T$ and all objective functions values must satisfy that $\sum_{i=1}^{m} (f_i)^2 = 1$. It is recommended to use $k = 10$.

DTLZ3

This problem is the same as DTLZ2 except for a new g function, that makes it multimodal. The definition is given as follows:

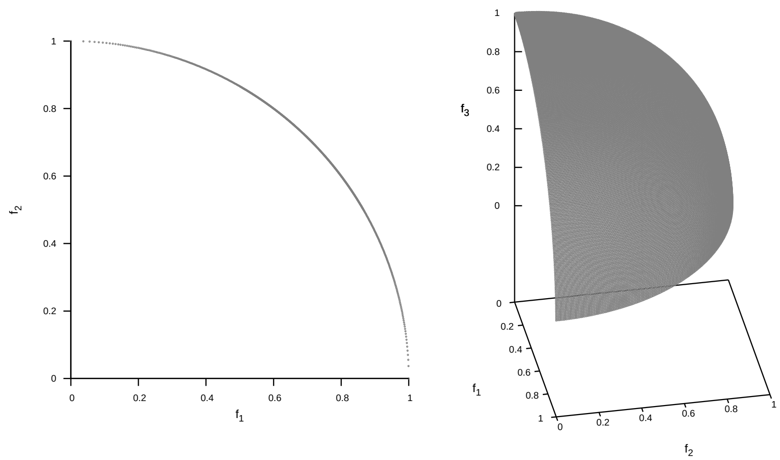


Figure A.2: DTLZ2 Pareto optimal front for two (left) and three (right) objectives.

$$
\begin{aligned}
\text{Given} \qquad & \vec{x} = \{x_1, \dots, x_{m-1}, x_m, \dots, x_n\} \\
\text{Minimize} \qquad & f_1(\vec{x}) = (1 + g(\vec{y})) \prod_{i=1}^{m-1} \cos(x_i \pi/2) \\
& f_{j=2:m-1}(\vec{x}) = (1 + g(\vec{y})) \left(\prod_{i=1}^{m-j} \cos(x_i \pi/2) \right) \sin(x_{m-j+1} \pi/2) \\
& f_m(\vec{x}) = (1 + g(\vec{y})) \sin(x_1 \pi/2) \\
\text{where} \qquad & y_{i=1:k} = \{x_m, x_{m+1}, \dots, x_n\} \\
& g(\vec{y}) = 100 \left[k + \sum_{i=1}^{k} (y_i - 0.5)^2 - \cos(20\pi(y_i - 0.5)) \right] \\
\text{subject to} \qquad & \forall i \in \{1, \dots, n\} \quad 0 \leq x_i \leq 1.
\end{aligned}
\tag{A.3}
$$

It is suggested that $k = 10$. The above function g introduces $(3^k - 1)$ local Pareto optimal fronts, and one global Pareto optimal front (see Figure A.3). All local Pareto optimal fronts are parallel to the global Pareto optimal front and an optimizer can get stuck at any of these local Pareto optimal fronts, before converging to the global Pareto optimal front at $g = 0$. The global Pareto optimal front corresponds to $\vec{y} = (0.5, 0.5, \dots)^T$. The next local Pareto optimal front is at $g = 1$.

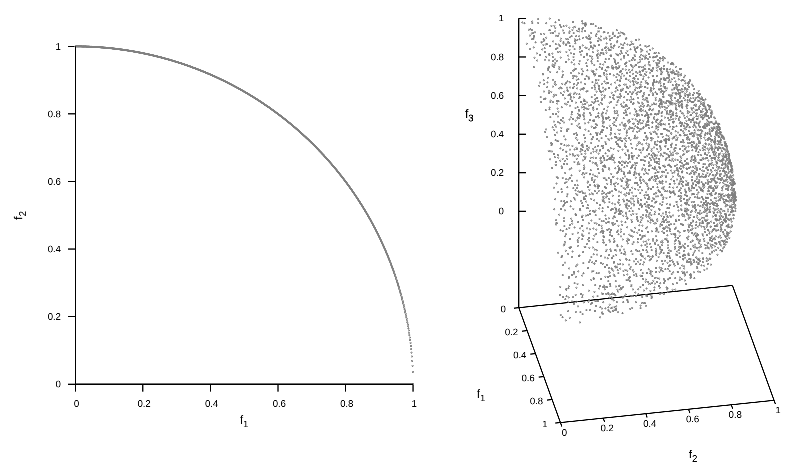


Figure A.3: DTLZ3 Pareto optimal front for two (left) and three (right) objectives.

DTLZ4

This problem is concave, separable and unimodal (see Figure A.4). It tests an optimizer's ability to maintain a good distribution of solutions, and is defined as follows:

$$
\begin{aligned}
&\text{Given} && \vec{x} = \{x_1, \ldots, x_{m-1}, x_m, \ldots, x_n\} \\
&\text{Minimize} && f_1(\vec{x}) = (1 + g(\vec{y})) \prod_{i=1}^{m-1} \cos(x_i^{\alpha} \pi/2) \\
& && f_{j=2:m-1}(\vec{x}) = (1 + g(\vec{y})) \left(\prod_{i=1}^{m-j} \cos(x_i^{\alpha} \pi/2) \right) \sin(x_{m-j+1}^{\alpha} \pi/2) \\
& && f_m(\vec{x}) = (1 + g(\vec{y})) \sin(x_1^{\alpha} \pi/2) \\
&\text{where} && y_{i=1:k} = \{x_m, x_{m+1}, \ldots, x_n\} \\
& && g(\vec{y}) = \sum_{i=1}^{k} (y_i - 0.5)^2 \\
&\text{subject to} && \forall i \in \{1, \ldots, n\} \quad 0 \le x_i \le 1.
\end{aligned} \tag{A.4}
$$

The parameters $\alpha = 100$ and $k = 10$ are suggested here. This problem allows a dense set of solutions to exist near the $f_m - f_1$ plane. It is interesting to note, that although the search space has a variable density of solutions, the classical weighted-sum approaches or other directional methods may not have any added difficulty in solving this problem compared to DTLZ2.

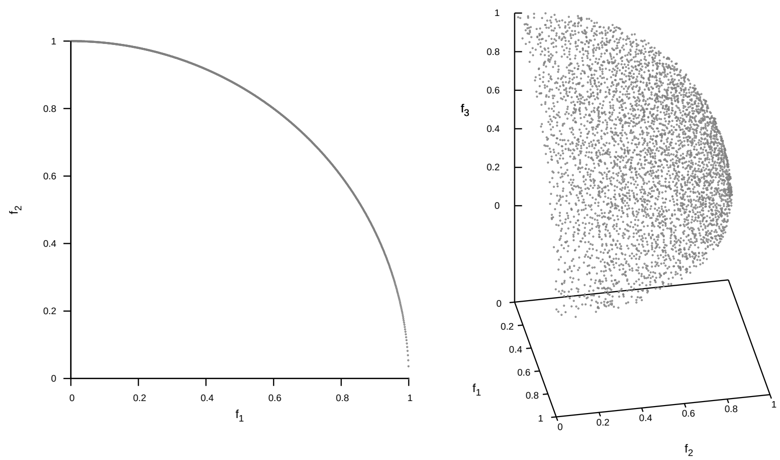


Figure A.4: DTLZ4 Pareto optimal front for two (left) and three (right) objectives.

DTLZ5

This problem is unimodal and degenerated (see Figure A.5). It is defined as:

$$
\begin{array}{lrl}
\text{Given} & \vec{x} = & \{x_1, \ldots, x_{m-1}, x_m, \ldots, x_n\} \\
\text{Minimize} & f_1(\vec{x}) = & (1 + g(\vec{y})) \prod_{i=1}^{m-1} \cos(\theta_i \pi/2) \\
& f_{j=2:m-1}(\vec{x}) = & (1 + g(\vec{y})) \left(\prod_{i=1}^{m-j} \cos(\theta_i \pi/2) \right) \sin(\theta_{m-j+1} \pi/2) \\
& f_m(\vec{x}) = & (1 + g(\vec{y})) \sin(\theta_1 \pi/2) \\
\text{where} & y_{i=1:k} = & \{x_m, x_{m+1}, \ldots, x_n\} \\
& \theta_i = & \begin{cases} x_i, & i = 1 \\ \frac{1+2g(\vec{y})}{2(1+g(\vec{y}))} x_i, & \forall i \in \{2, 3, \ldots, m-1\} \end{cases} \\
& g(\vec{y}) = & \sum_{i=1}^{k} (y_i - 0.5)^2 \\
\text{subject to} & & \forall i \in \{1, \ldots, n\} \quad 0 \leq x_i \leq 1.
\end{array}
\tag{A.5}
$$

This problem will test an optimizer's ability to converge to a curve. The g function with $k = 10$ variables is suggested. The Pareto optimal front corresponds to $\vec{y} = (0.5, 0.5, \ldots)^T$, and all objective function values must satisfy $\sum_{i=1}^{m} (f_i)^2 = 1$.

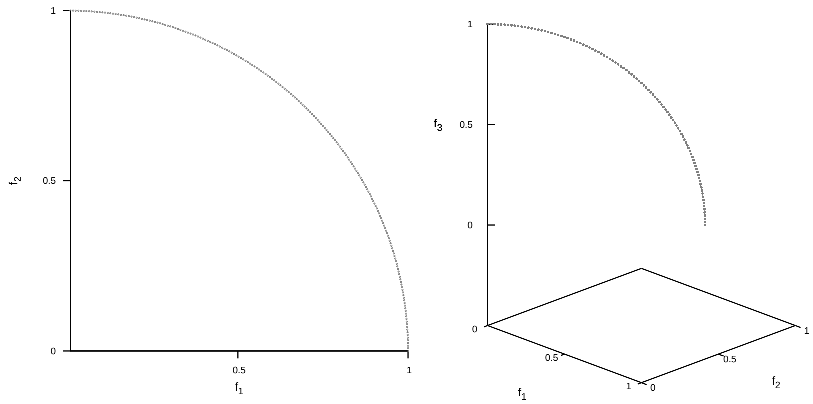


Figure A.5: DTLZ5 Pareto optimal front for two (left) and three (right) objectives.

DTLZ6

Modifying DTLZ5, a harder problem evolves by changing the function g. The resulting problem is unimodal and degenerated:

$$
\begin{aligned}
&\text{Given} && \vec{x} = \{x_1, \ldots, x_{m-1}, x_m, \ldots, x_n\} \\
&\text{Minimize} && f_1(\vec{x}) = (1 + g(\vec{y})) \prod_{i=1}^{m-1} \cos(\theta_i \pi/2) \\
& && f_{j=2:m-1}(\vec{x}) = (1 + g(\vec{y})) \left(\prod_{i=1}^{m-j} \cos(\theta_i \pi/2) \right) \sin(\theta_{m-j+1}\pi/2) \\
& && f_m(\vec{x}) = (1 + g(\vec{y})) \sin(\theta_1 \pi/2) \\
&\text{where} && y_{i=1:k} = \{x_m, x_{m+1}, \ldots, x_n\} \\
& && \theta_i = \begin{cases} x_i, & i = 1 \\ \frac{1+2g(\vec{y})}{2(1+g(\vec{y}))} x_i, & \forall i \in \{2, 3, \ldots, m-1\} \end{cases} \\
& && g(\vec{y}) = \sum_{i=1}^{k} y_i^{0.1} \\
&\text{subject to} && \forall i \in \{1, \ldots, n\} \quad 0 \leq x_i \leq 1.
\end{aligned}
\tag{A.6}
$$

The Pareto optimal front corresponds to $\vec{y} = (0, 0, \ldots)^T$ and is shown in Figure A.6. The value of k is chosen as 10. The lack of convergence to the Pareto

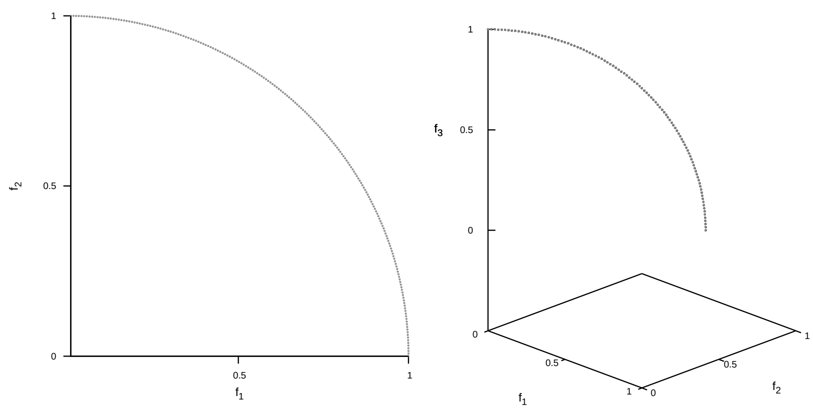


Figure A.6: DTLZ6 Pareto optimal front for two (left) and three (right) objectives.

optimal front in this problem makes optimizers to find a dominated surface as the obtained front, whereas the true Pareto optimal front is a curve. In real-world problems, this aspect may provide misleading information about the properties of the Pareto optimal front.

DTLZ7

This problem has a disconnected set of 2^{m-1} Pareto optimal regions in the search space and will test an algorithm's ability to maintain subpopulations in different Pareto optimal regions.

$$
\begin{aligned}
&\text{Given} && \vec{x} = \{x_1, \ldots, x_{m-1}, x_m, \ldots, x_n\} \\
&\text{Minimize} && f_{j=1:m-1}(\vec{x}) = x_j \\
& && f_m(\vec{x}) = (1+g(\vec{y}))\left(m - \sum_{i=1}^{m-1}\left[\frac{f_i}{1+g(\vec{y})}\left(1+\sin(3\pi f_i)\right)\right]\right) \\
&\text{where} && y_{i=1:k} = \{x_m, x_{m+1}, \ldots, x_n\} \\
& && g(\vec{y}) = 1 + \frac{9}{k}\sum_{i=1}^{k} y_i \\
&\text{subject to} && \forall i \in \{1, \ldots, n\} \quad 0 \leq x_i \leq 1.
\end{aligned}
\tag{A.7}
$$

The function g requires $k = 20$ decision variables. The Pareto optimal solutions correspond to $\vec{y} = (0, 0, \ldots)^T$. The Pareto optimal front is shown in Figure A.7.

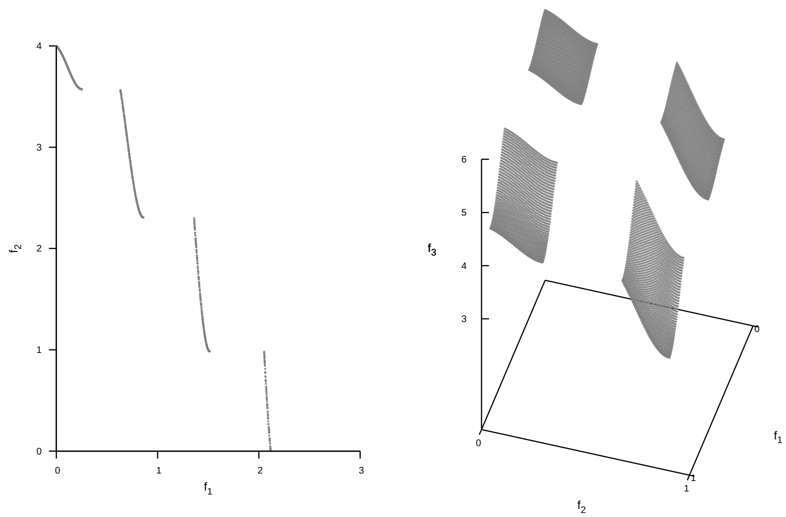


Figure A.7: DTLZ7 Pareto optimal front for two (left) and three (right) objectives.

A.2 Walking Fish Group Test Suite

The Walking-Fish-Group test suite, published in 2005 by Huband et al. [48], suggests nine multi-objective test problems (WFG1-WFG9), that are scalable with respect to both objectives and variables, and have known Pareto optimal sets. These problems include a wide variety of Pareto optimal geometries. Moreover, characteristics such as bias, multi-modality, and non-separability are defined by a set of transformations.

In the following, we present these benchmark problems. Here, m represents the number of objectives, and each problem is defined in terms of an underlying vector of parameters $\vec{x} \in \mathbb{R}^m$ that defines the fitness space. All $x_i \in \vec{x}$ will have domain $[0, 1]$. x_m is known as the underlying distance parameter, and $x_{1:m-1}$ are the underlying position parameters. The vector $\vec{x}$ is derived, via a series of transition vectors, from a vector of working parameters $\vec{z} \in \mathbb{R}^n$ (also known as vector of variables). The domain of all $z_i \in \vec{z}$ is $[0, 2i]$. It is worth noting, that $n \geq m$ and $n = k + l$. The first $k \in \{m-1, 2(m-1), 3(m-1), \ldots\}$ working parameters are the position related parameters and the last $l \in \{1, 2, \ldots\}$ working parameters are the distance related parameters. Each transition vector adds complexity to the underlying problem. The optimizer directly manipulates $\vec{z}$, through which $\vec{x}$ is indirectly manipulated.

WFG1

This problem is separable and unimodal, but it has a polynomial and flat region. It is strongly biased toward small values of the variables, which makes it very difficult for some optimizers. It is defined as follows:

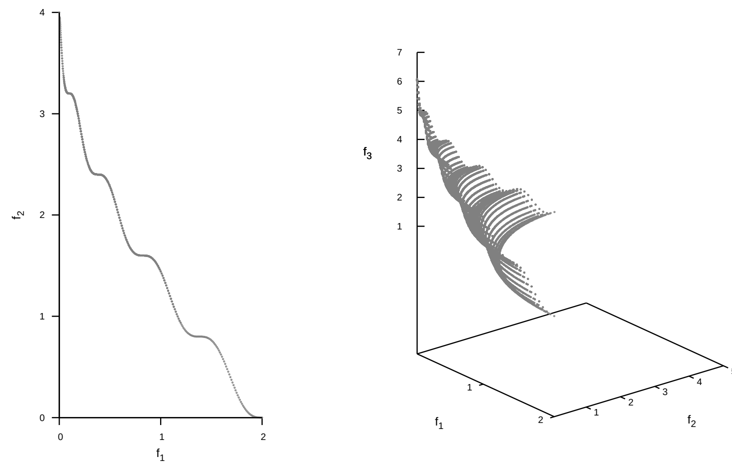


Figure A.8: WFG1 Pareto optimal front for two (left) and three (right) objectives.

$$
\begin{aligned}
\text{Given} \quad & \vec{z} = \{z_1, \ldots, z_k, z_{k+1}, \ldots, z_n\} \\
\text{Minimize} \quad & f_1(\vec{x}) = x_m + 2\prod_{i=1}^{m-1}(1-\cos(x_i\pi/2)) \\
& f_{j=2:m-1}(\vec{x}) = x_m + 2j\left(\prod_{i=1}^{m-j}(1-\cos(x_i\pi/2))\right)(1-\sin(x_{m-j+1}\pi/2)) \\
& f_m(\vec{x}) = x_m + 2m\left(1 - x_1 - \frac{\cos(10\pi x_1 + \pi/2)}{10\pi}\right) \\
\text{where} \quad & x_{i=1:m-1} = \text{r\_sum}(\left\{y_{(i-1)k/(m-1)+1}, \ldots, y_{ik/(m-1)}\right\}, \\
& \qquad \{2(i-1)k/(m-1)+1, \ldots, 2ik/(m-1)\}) \\
& x_m = \text{r\_sum}(\{y_{k+1}, \ldots, y_n\}, \{2(k+1), \ldots, 2n\}) \\
& y_{i=1:n} = \text{b\_poly}(y'_i, 0.02) \\
& y'_{i=1:k} = y''_i \\
& y'_{i=k+1:n} = \text{b\_flat}(y''_i, 0.8, 0.75, 0.85) \\
& y''_{i=1:k} = z_i/(2i) \\
& y''_{i=k+1:n} = \text{s\_linear}(z_i/(2i), 0.35)
\end{aligned}
\tag{A.8}
$$

The Pareto optimal front of the WFG1 test problem is shown in Figure A.8.

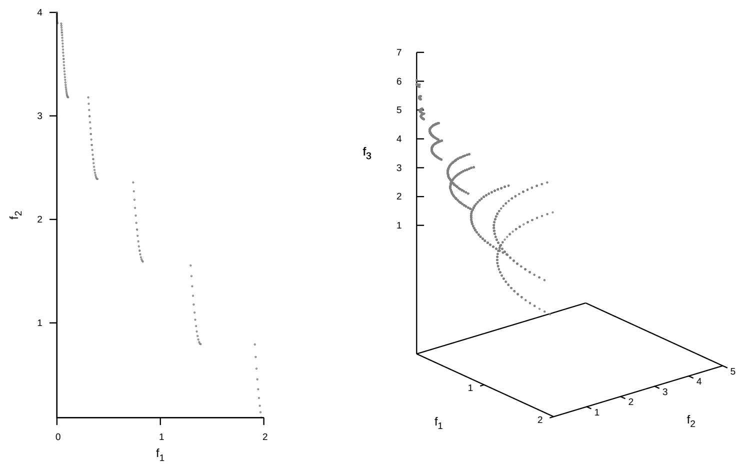


Figure A.9: WFG2 Pareto optimal front for two (left) and three (right) objectives.

WFG2

This problem is nonseparable and multimodal. The Pareto optimal front is disconnected (see Figure A.9), and is given by the following expression:

$$
\begin{aligned}
\text{Given} \quad & \vec{z} = \{z_1, \ldots, z_k, z_{k+1}, \ldots, z_n\} \\
\text{Minimize} \quad & f_1(\vec{x}) = x_m + 2 \prod_{i=1}^{m-1} (1 - \cos(x_i \pi/2)) \\
& f_{j=2:m-1}(\vec{x}) = x_m + 2j \left(\prod_{i=1}^{m-j} (1 - \cos(x_i \pi/2)) \right) (1 - \sin(x_{m-j+1} \pi/2)) \\
& f_m(\vec{x}) = x_m + 2m \left(1 - x_1 \cos^2(5 x_1 \pi)\right) \\
\text{where} \quad & x_{i=1:m-1} = \text{r\_sum}(\left\{y_{(i-1)k/(m-1)+1}, \ldots, y_{ik/(m-1)}\right\}, \{1, \ldots, 1\}) \\
& x_m = \text{r\_sum}(\left\{y_{k+1}, \ldots, y_{k+l/2}\right\}, \{1, \ldots, 1\}) \\
& y_{i=1:k} = y'_i \\
& y_{i=k+1:k+l/2} = \text{r\_nonsep}(\left\{y'_{k+2(i-k)-1}, y'_{k+2(i-k)}\right\}, 2) \\
& y'_{i=1:k} = z_i/(2i) \\
& y'_{i=k+1:n} = \text{s\_linear}(z_i/(2i), 0.35)
\end{aligned}
\tag{A.9}
$$

WFG3

This problem is nonseparable but unimodal. It has a linear and degenerated Pareto optimal front,[2] which is given by the following expression:

$$
\begin{aligned}
\text{Given} \quad & \vec{z} = \{z_1, \ldots, z_k, z_{k+1}, \ldots, z_n\} \\
\text{Minimize} \quad & f_1(\vec{x}) = x_m + 2 \prod_{i=1}^{m-1} x_i \\
& f_{j=2:m-1}(\vec{x}) = x_m + 2j \left(\prod_{i=1}^{m-j} x_i \right) (1 - x_{m-j+1}) \\
& f_m(\vec{x}) = x_m + 2m\,(1 - x_1) \\
\text{where} \quad & x_{i=1} = u_i \\
& x_{i=2:m-1} = x_m\,(u_i - 0.5) + 0.5 \\
& x_m = \text{r\_sum}(\{y_{k+1}, \ldots, y_{k+l/2}\}, \{1, \ldots, 1\}) \\
& u_i = \text{r\_sum}(\{y_{(i-1)k/(m-1)+1}, \ldots, y_{ik/(m-1)}\}, \{1, \ldots, 1\}) \\
& y_{i=1:k} = y'_i \\
& y_{i=k+1:k+l/2} = \text{r\_nonsep}(\{y'_{k+2(i-k)-1}, y'_{k+2(i-k)}\}, 2) \\
& y'_{i=1:k} = z_i/(2i) \\
& y'_{i=k+1:n} = \text{s\_linear}(z_i/(2i), 0.35)
\end{aligned}
\tag{A.10}
$$

In Figure A.10, the Pareto optimal front is represented for 2D and 3D, respectively.

WFG4

In this case, the problem is separable, but highly multimodal. The Pareto optimal front is concave (see Figure A.11) and is defined as follows:

$$
\begin{aligned}
\text{Given} \quad & \vec{z} = \{z_1, \ldots, z_k, z_{k+1}, \ldots, z_n\} \\
\text{Minimize} \quad & f_1(\vec{x}) = x_m + 2 \prod_{i=1}^{m-1} \sin(x_i\pi/2) \\
& f_{j=2:m-1}(\vec{x}) = x_m + 2j \left(\prod_{i=1}^{m-j} \sin(x_i\pi/2) \right) \cos(x_{m-j+1}\pi/2) \\
& f_m(\vec{x}) = x_m + 2m\cos(x_1\pi/2) \\
\text{where} \quad & x_{i=1:m-1} = \text{r\_sum}(\{y_{(i-1)k/(m-1)+1}, \ldots, y_{ik/(m-1)}\}, \{1, \ldots, 1\}) \\
& x_m = \text{r\_sum}(\{y_{k+1}, \ldots, y_n\}, \{1, \ldots, 1\}) \\
& y_{i=1:n} = \text{s\_multi}(z_i/(2i), 30, 10, 0.35)
\end{aligned}
\tag{A.11}
$$

[2] The dimensionality of the Pareto optimal front is $m - 2$.

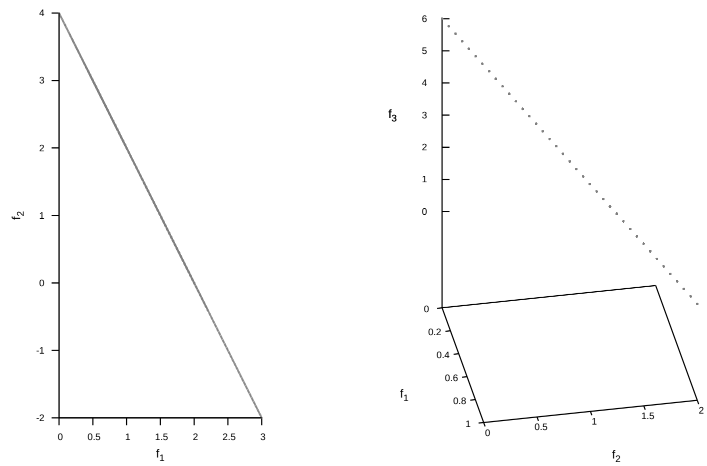


Figure A.10: WFG3 Pareto optimal front for two (left) and three (right) objectives.

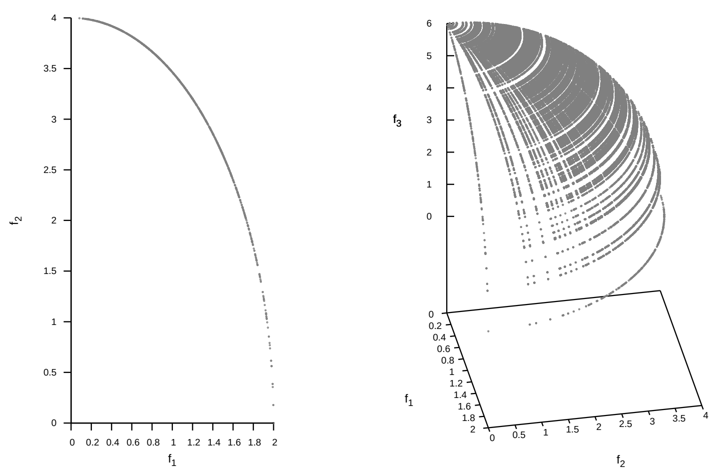


Figure A.11: WFG4 Pareto optimal front for two (left) and three (right) objectives.

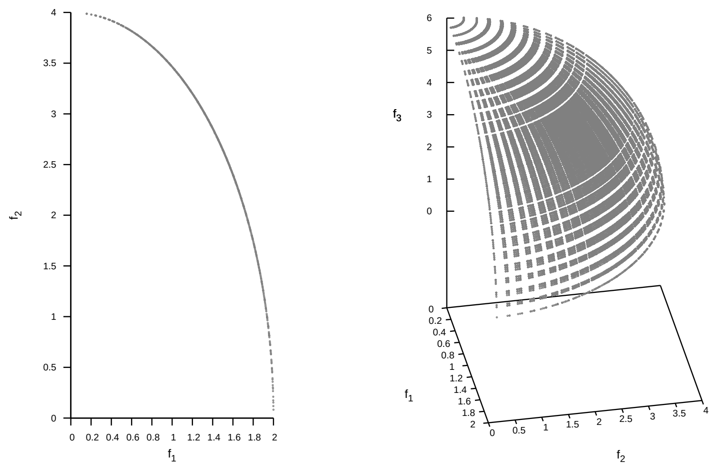


Figure A.12: WFG5 Pareto optimal front for two (left) and three (right) objectives.

WFG5

A deceptive and separable problem. The Pareto optimal front is concave (see Figure A.12) and is defined by the following expression:

$$
\begin{aligned}
\text{Given} \quad & \vec{z} = \{z_1, \ldots, z_k, z_{k+1}, \ldots, z_n\} \\
\text{Minimize} \quad & f_1(\vec{x}) = x_m + 2 \prod_{i=1}^{m-1} \sin(x_i \pi/2) \\
& f_{j=2:m-1}(\vec{x}) = x_m + 2j \left(\prod_{i=1}^{m-j} \sin(x_i \pi/2) \right) \cos(x_{m-j+1} \pi/2) \\
& f_m(\vec{x}) = x_m + 2m \cos(x_1 \pi/2) \\
\text{where} \quad & x_{i=1:m-1} = \text{r\_sum}(\left\{ y_{(i-1)k/(m-1)+1}, \ldots, y_{ik/(m-1)} \right\}, \{1, \ldots, 1\}) \\
& x_m = \text{r\_sum}(\{y_{k+1}, \ldots, y_n\}, \{1, \ldots, 1\}) \\
& y_{i=1:n} = \text{s\_decept}(z_i/(2i), 0.35, 0.001, 0.05)
\end{aligned}
\tag{A.12}
$$

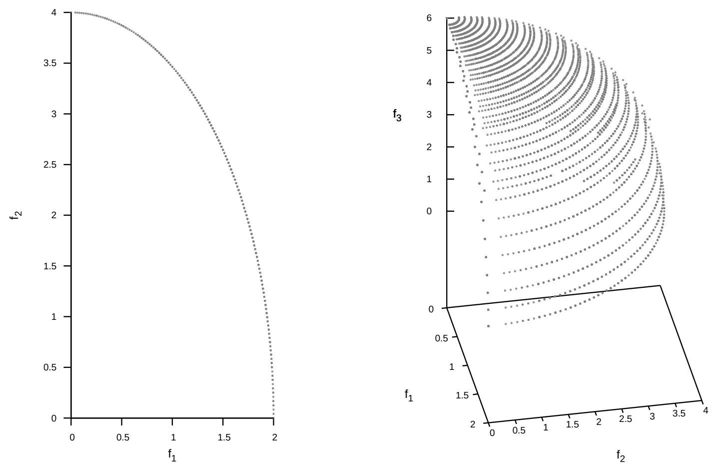


Figure A.13: WFG6 Pareto optimal front for two (left) and three (right) objectives.

WFG6

This problem is nonseparable and unimodal. Its Pareto optimal front is concave (see Figure A.13), and is defined as follows:

$$\begin{aligned}
\text{Given} \quad & \vec{z} = \{z_1, \ldots, z_k, z_{k+1}, \ldots, z_n\} \\
\text{Minimize} \quad & f_1(\vec{x}) = x_m + 2 \prod_{i=1}^{m-1} \sin(x_i\pi/2) \\
& f_{j=2:m-1}(\vec{x}) = x_m + 2j \left(\prod_{i=1}^{m-j} \sin(x_i\pi/2) \right) \cos(x_{m-j+1}\pi/2) \\
& f_m(\vec{x}) = x_m + 2m \cos(x_1\pi/2) \\
\text{where} \quad & x_{i=1:m-1} = \text{r_nonsep}(\left\{y_{(i-1)k/(m-1)+1}, \ldots, y_{ik/(m-1)}\right\}, k/(m-1)) \\
& x_m = \text{r_nonsep}(\{y_{k+1}, \ldots, y_n\}, l) \\
& y_{i=1:k} = z_i/(2i) \\
& y_{i=k+1:n} = \text{s_linear}(z_i/(2i), 0.35)
\end{aligned} \quad \text{(A.13)}$$

WFG7

Having a parameter dependent bias, this problem is also separable and unimodal. The concave Pareto optimal front is depicted in Figure A.14, and is defined as:

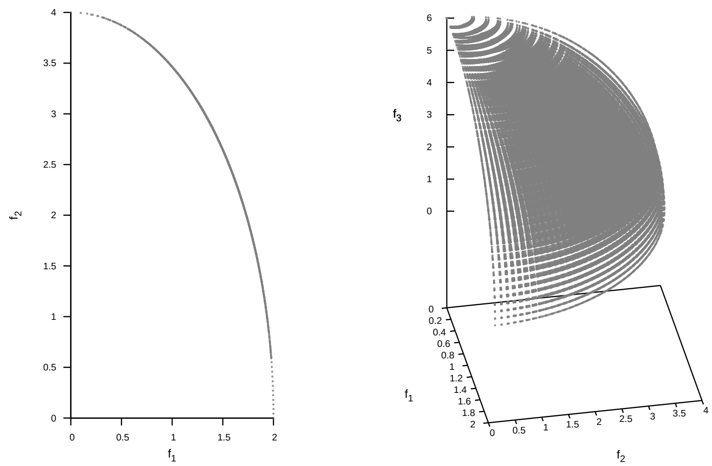


Figure A.14: WFG7 Pareto optimal front for two (left) and three (right) objectives.

$$
\begin{aligned}
\text{Given} \qquad & \vec{z} = \{z_1, \ldots, z_k, z_{k+1}, \ldots, z_n\} \\
\text{Minimize} \qquad f_1(\vec{x}) &= x_m + 2\prod_{i=1}^{m-1} \sin(x_i\pi/2) \\
f_{j=2:m-1}(\vec{x}) &= x_m + 2j\left(\prod_{i=1}^{m-j} \sin(x_i\pi/2)\right)\cos(x_{m-j+1}\pi/2) \\
f_m(\vec{x}) &= x_m + 2m\cos(x_1\pi/2) \\
\text{where} \qquad x_{i=1:m-1} &= \text{r\_sum}(\left\{y_{(i-1)k/(m-1)+1}, \ldots, y_{ik/(m-1)}\right\}, \{1, \ldots, 1\}) \\
x_m &= \text{r\_sum}(\{y_{k+1}, \ldots, y_n\}, \{1, \ldots, 1\}) \\
y_{i=1:k} &= y'_i \\
y_{i=k+1:n} &= \text{s\_linear}(y'_i, 0.35) \\
y'_{i=1:k} &= \text{b\_param}(z_i/(2i), \text{r\_sum}(\{z_{i+1}/(2(i+1)), \ldots, z_n/(2n)\}, \\
&\quad \{1, \ldots, 1\}), \frac{0.98}{49.98}, 0.02, 50) \\
y'_{i=k+1:n} &= z_i/(2i)
\end{aligned}
\tag{A.14}
$$

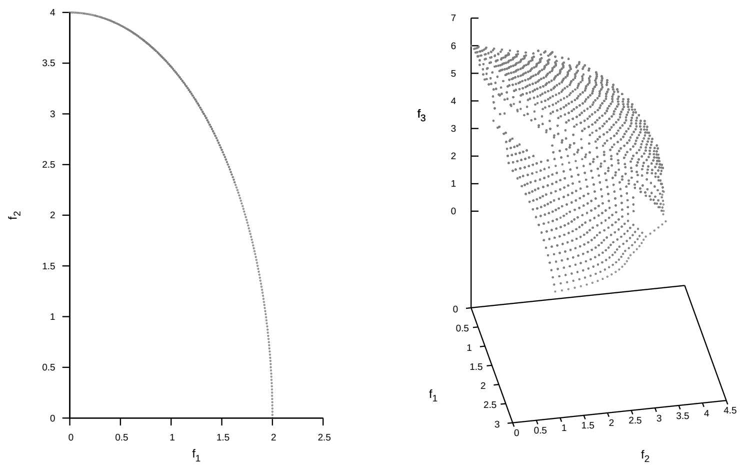


Figure A.15: WFG8 Pareto optimal front for two (left) and three (right) objectives.

WFG8

This problem also has a parameter dependent bias, but is also nonseparable and unimodal. The concave Pareto optimal front (see Figure A.15) is given by the expression:

$$
\begin{aligned}
\text{Given} \quad & \vec{z} = \{z_1, \ldots, z_k, z_{k+1}, \ldots, z_n\} \\
\text{Minimize} \quad & f_1(\vec{x}) = x_m + 2 \prod_{i=1}^{m-1} \sin(x_i \pi/2) \\
& f_{j=2:m-1}(\vec{x}) = x_m + 2j \left(\prod_{i=1}^{m-j} \sin(x_i \pi/2) \right) \cos(x_{m-j+1}\pi/2) \\
& f_m(\vec{x}) = x_m + 2m \cos(x_1 \pi/2) \\
\text{where} \quad & x_{i=1:m-1} = \text{r\_sum}(\left\{ y_{(i-1)k/(m-1)+1}, \ldots, y_{ik/(m-1)} \right\}, \{1, \ldots, 1\}) \\
& x_m = \text{r\_sum}(\{y_{k+1}, \ldots, y_n\}, \{1, \ldots, 1\}) \\
& y_{i=1:k} = y'_i \\
& y_{i=k+1:n} = \text{s\_linear}(y'_i, 0.35) \\
& y'_{i=k} = z_i/(2i) \\
& y'_{i=k+1:n} = \text{b\_param}(z_i/(2i), \text{r\_sum}(\{z_1/2, \ldots, z_{i-1}/(2(i-1))\}, \\
& \qquad \{1, \ldots, 1\}), \frac{0.98}{49.98}, 0.02, 50)
\end{aligned} \tag{A.15}
$$

WFG9

The last problem of the suite is nonseparable, multimodal, deceptive, and has a parameter dependent bias. All these features make it a very difficult problem. The concave Pareto optimal front (see Figure A.16) is defined as:

$$
\begin{aligned}
&\text{Given} && \vec{z} = \{z_1, \ldots, z_k, z_{k+1}, \ldots, z_n\} \\
&\text{Minimize} & f_1(\vec{x}) &= x_m + 2 \prod_{i=1}^{m-1} \sin(x_i \pi/2) \\
& & f_{j=2:m-1}(\vec{x}) &= x_m + 2j \left(\prod_{i=1}^{m-j} \sin(x_i \pi/2) \right) \cos(x_{m-j+1}\pi/2) \\
& & f_m(\vec{x}) &= x_m + 2m \cos(x_1 \pi/2) \\
&\text{where} & x_{i=1:m-1} &= \text{r\_nonsep}(\left\{y_{(i-1)k/(m-1)+1}, \ldots, y_{ik/(m-1)}\right\}, k/(m-1)) \\
& & x_m &= \text{r\_nonsep}(\{y_{k+1}, \ldots, y_n\}, l) \\
& & y_{i=1:k} &= \text{s\_decept}(y'_i, 0.35, 0.001, 0.05) \\
& & y_{i=k+1:n} &= \text{s\_multi}(y'_i, 30, 95, 0.35) \\
& & y'_{i=1:n-1} &= \text{b\_param}(z_i/(2i), \text{r\_sum}(\{z_{i+1}/(2(i+1)), \ldots, z_n/(2n)\}, \\
& & &\quad \{1, \ldots, 1\}), \frac{0.98}{49.98}, 0.02, 50) \\
& & y'_n &= z_n/(2n)
\end{aligned}
\tag{A.16}
$$

For WFG1-WFG7, a solution is Pareto optimal if $z_{i=k+1:n} = (2i)0.35$. Note that WFG2 is disconnected. For WFG8, it is required that all of:

$$
\begin{aligned}
z_{i=k+1:n} &= (2i)0.35^{\left(0.02+49.98\left(\frac{0.98}{49.98}-(1-2u)|\lfloor 0.5-u \rfloor+\frac{0.98}{49.98}|\right)\right)^{-1}}, \\
u &= \text{r\_sum}(\{z_1, \ldots, z_{i-1}\}, \{1, \ldots, 1\}).
\end{aligned}
$$

To obtain a Pareto optimal solution, the position should first be determined by setting $z_{1:k}$ appropriately. The required distance-related parameter values can then be calculated by first determining z_{k+1}, then z_{k+2}, and so on, until z_n has been calculated.

In the case of WFG9, for a solution to be Pareto optimal, it is required that all of:

$$
z_{i=k+1:n} = (2i) \begin{cases} 0.35^{(0.02+1.96\ \text{r\_sum}(\{z_{i+1},\ldots,z_n\},\{1,\ldots,1\}))^{-1}}, & i \neq n \\ 0.35, & i = n, \end{cases}
$$

which can be found by first determining z_n, then z_{n-1}, and so on, until the required value for z_{k+1} is determined. Once the optimal values for $z_{k+1:n}$ are determined, the position-related parameters can be varied arbitrarily to obtain different Pareto optimal solutions.

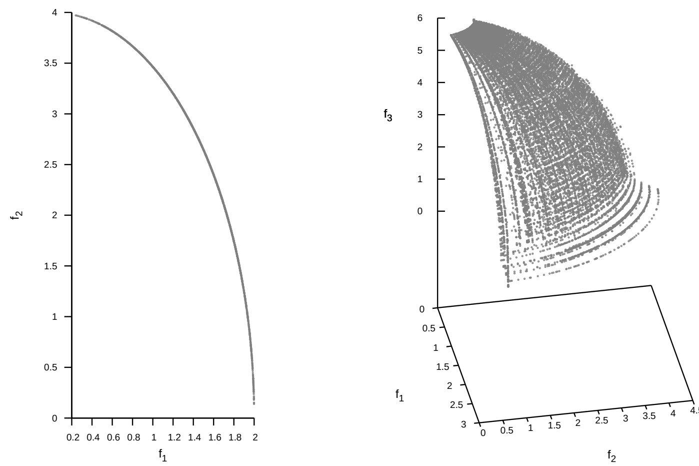


Figure A.16: WFG9 Pareto optimal front for two (left) and three (right) objectives.

Transformation Functions

The previous problems are defined in terms of a set of transformation functions, which map parameters with domain $[0, 1]$ onto the range $[0, 1]$. There are three types of transformation functions: *bias, shift and reduction functions.* Bias and shift functions only employ one parameter, whereas reduction functions can employ many. Bias transformations have a natural impact on the search process by biasing the fitness landscape. Shift transformations move the location of optimal values, and are used to apply a linear shift, or to produce deceptive and multimodal problems. Reduction transformations are used to produce non-separability of the problem (dependency between variables). In the following, we define such transformation functions.

Bias: Polynomial

When $\alpha > 1$ or when $\alpha < 1$, y is biased towards zero or towards one, respectively.

$$\text{b\_poly}(y, \alpha) = y^{\alpha}, \tag{A.17}$$

where $\alpha > 0$ and $\alpha \neq 1$.

Bias: Flat Region

Values of y between B and C, the area of the flat region, are mapped to the value A.

$$\begin{aligned}\text{b\_flat}(y, A, B, C) = A + \min(0, \lfloor y - B \rfloor)\frac{A(B-y)}{B} \\ - \min(0, \lfloor C - y \rfloor)\frac{(1-A)(y-C)}{1-C},\end{aligned} \tag{A.18}$$

where $A, B, C \in [0, 1]$, $B < C$, $B = 0 \Rightarrow A = 0 \wedge C \neq 1$, and $C = 1 \Rightarrow A = 1 \wedge B \neq 0$.

Bias: Parameter Dependent

A, B, C, the parameter vector $\vec{w} \in [0, 1]^{|\vec{w}|}$, and the reduction function u together determine the degree to which y is biased by being raised to an associated power: values of $u(\vec{w}) \in [0, 0.5]$ are mapped linearly onto $[B, B + (C - B)A]$, and values of $u(\vec{w}) \in [0.5, 1]$ are mapped linearly onto $[B + (C - B)A, C]$.

$$\text{b\_param}(y, u(\vec{w}), A, B, C) = y^{B+(C-B)(A-(1-2u(\vec{w}))|\lfloor 0.5-u(\vec{w})\rfloor+A|)}, \tag{A.19}$$

where $A \in (0, 1)$, and $0 < B < C$.

Shift: Linear

$A \in (0, 1)$ is the value for which y is mapped to zero.

$$\text{s\_linear}(y, A) = \frac{|y - A|}{|\lfloor A - y \rfloor + A|}. \tag{A.20}$$

Shift: Deceptive

A is the value at which y is mapped to zero, and the global minimum of the transformation. B is the "aperture" size of the well/basin leading to the global minimum at A, and C is the value of the deceptive minima (there are always two deceptive minima).

$$\begin{aligned}\text{s\_decept}(y, A, B, C) = 1 + (|y - A| - B)\Bigg(\frac{\lfloor y - A + B \rfloor \left(1 - C + \frac{A-B}{B}\right)}{A - B} \\ + \frac{\lfloor A + B - y \rfloor \left(1 - C + \frac{1-A-B}{B}\right)}{1 - A - B} + \frac{1}{B}\Bigg),\end{aligned} \tag{A.21}$$

where $A \in (0, 1)$, $0 < B \ll 1$, $0 < C \ll 1$, $A - B > 0$, and $A + B < 1$.

Shift: Multi-modal

A controls the number of minima, B controls the magnitude of the "hill sizes" of the multimodality, and C is the value for which y is mapped to zero. When $B = 0$, $2A + 1$ values of y (one at C) are mapped to zero, and when $B \neq 0$, there are $2A$ local minima, and one global minimum at C. Larger values of A and smaller values of B create more difficult problems.

$$\text{s\_multy}(y, A, B, C) = \frac{1 + \cos\left[(4A + 2)\,\pi\left(0.5 - \frac{|y-C|}{2(\lfloor C-y \rfloor + C)}\right)\right] + 4B\left(\frac{|y-C|}{2(\lfloor C-y \rfloor + C)}\right)^2}{B + 2}, \tag{A.22}$$

where $A \in \{1, 2, \ldots\}$, $B \geq 0$, $(4A + 2)\pi > 4B$, and $C \in (0, 1)$.

Reduction: Weighted Sum

By varying the constants of the weight vector $\vec{w}$, optimizers can be forced to treat parameters differently.

$$\text{r\_sum}(\vec{y}, \vec{w}) = \frac{\left(\sum_{i=1}^{|\vec{y}|} w_i y_i\right)}{\sum_{i=1}^{|\vec{y}|} w_i}, \tag{A.23}$$

where $|w| = |y|$, and $w_1, \ldots, w_{|\vec{y}|>0}$.

Reduction: Non-separable

A controls the degree of non-separability.[3]

$$\text{r\_nonsep}(\vec{y}, A) = \frac{\sum_{j=1}^{|\vec{y}|}\left(y_j + \sum_{k=0}^{A-2} |y_j - y_{1+(j+k)mod|\vec{y}|}|\right)}{\frac{|\vec{y}|}{A}\lceil A/2 \rceil\left(1 + 2A - 2\lceil A/2 \rceil\right)}, \tag{A.24}$$

where $A \in \{1, \ldots, |\vec{y}|\}$, and $|\vec{y}| \text{mod} A = 0$.

A.3 Summary

When attempting to better understand the strengths and weaknesses of an optimizer, it is important to have a strong understanding of the problem at hand. For this reason, a large set of artificial test problems have been proposed.

In this appendix, we reviewed two important benchmarks in the field of many-objective optimization: the DTLZ test suite, which includes seven unconstrained problems with degenerate and multimodal Pareto optimal fronts, and the WFG test suite, which includes nine problems with a wide variety of Pareto optimal geometries and fitness landscape, such as, bias, multi-modality, and non-separability.

[3] It is worth noting that, $\text{r\_nonsep}(\vec{y}, 1) = \text{r\_sum}(\vec{y}, \{1, \ldots, \})$.

Both benchmarks are scalable with respect to the number of objectives and variables. In the case of the DTLZ test suite, the exact location of the Pareto optimal sets are known, as well as the exact shapes and locations of almost all their corresponding Pareto optimal fronts. On the other hand, for the WFG test suite, only the Pareto optimal sets are known.

Therefore, information in objective space can be used by performance indicators that measure convergence to the true Pareto optimal front, such as, generational distance, inverse generational distance, etc., allowing to perform a statistical analysis of results.

There is some controversy about the DTLZ test suite, since it has several limitations, such as the following:

- None of its problems is deceptive.
- None of its problems is (practically) nonseparable.
- The number of position parameters is always fixed, relative to the number of objectives.
- DTLZ5 y DTLZ6 are both meant to be problems with degenerate Pareto optimal fronts. However, this is untrue for instances with four or more objectives. Additionally, their Pareto optimal fronts are unclear beyond three objectives.

Because of this, the WFG test suite is a better choice, since it relies on a set of transformations, that cover the missing features of the DTLZ test suite. In consequence, it is more difficult to solve for optimizers, specially for those that are based on hill climbing strategies.

As a final comment, these benchmarks were suggested in order to cover representative cases. However, the user can design their own problems following the methodology described in [214, 48].

Appendix B

Statistics Applied to Stochastic Optimizers

One important task in Evolutionary Computation is the analysis of optimizers in terms of efficiency, i.e., the computational effort required (CPU time, memory, number of evaluations/iterations) and effectiveness, which can be measured in terms of convergence, distribution and accuracy of the obtained solutions. One may wish to determine if a given MOEA performs "better" than another over a specific problem domain class or classes, and if so, we also wish to determine the reason for that.

The proper way to compare efficiency is by analyzing the complexity of the algorithms, and then, corroborate this theoretical result experimentally. On the other hand, the traditional modus operandi when comparing effectiveness is by generating random samples and then applying transformations that can allow us to describe and make inferences about the approximations produced by MOEAs. However, we cannot determine this fact definitively because we only have access to finite-sized samples of approximation sets. Instead, we should show these inferences statistically.

The aim of this Appendix is to present the adopted methodology for the analysis of MOEAs made in Chapter 5, which is based on nonparametric statistics.[1] In Section B.1, we present some basic concepts about probability theory and statistical inference, in order to understand the methodology described in Section B.2. We also provide a guide to the use of the most appropriate inferential test, as well as some of the most popular software packages currently available. Then, in Section B.3, we introduce the Wilcoxon Rank Sum Test, which is a powerful nonparametric technique that can compare two algorithms under certain assumptions. To make it easier to understand, we give an illustrative example of its use. Finally, in Section B.4 we present the summary of the Appendix.

[1] A statistical method wherein the data is not required to fit a normal distribution. Nonparametric statistics use data that is often ordinal, which means that it does not rely on numbers, but rather on a ranking or order of sorts.

B.1 Basic Definitions

In the following, we provide some basic terminology of probability theory and statistical inference, that will be employed in subsequent sections. If the reader is familiar with these topics, he/she may skip this subsection.

Definition B.1.1. A *sample space* is the set of all possible outcomes of a random experiment.

Definition B.1.2. A *random variable* X is a function that assigns real numbers to the points in a sample space.

Definition B.1.3. The *probability function of the random variable* X, usually denoted by $f(x)$, is the function that gives the probability of X assuming the value x, for any real number x. In other words:

$$f(x) = P(X = x). \tag{B.1}$$

Definition B.1.4. The *distribution function of a random variable* X, usually denoted by $F(X)$, is the function that gives the probability of X being less than or equal to any real number x. In other words:

$$F(x) = P(X \leq x) = \sum_{t \leq x} f(t), \tag{B.2}$$

where the summation extends over all values of t that do not exceed x.

If the graph of a distribution function has no steps but rises gradually, then the distribution function is called *continuous*, and the random variable with that distribution function is called a *continuous random variable*.

Definition B.1.5. The *expected value* (or, synonymously, *expectation* or *mean*) of a random variable X with probability function $f(x)$, is:

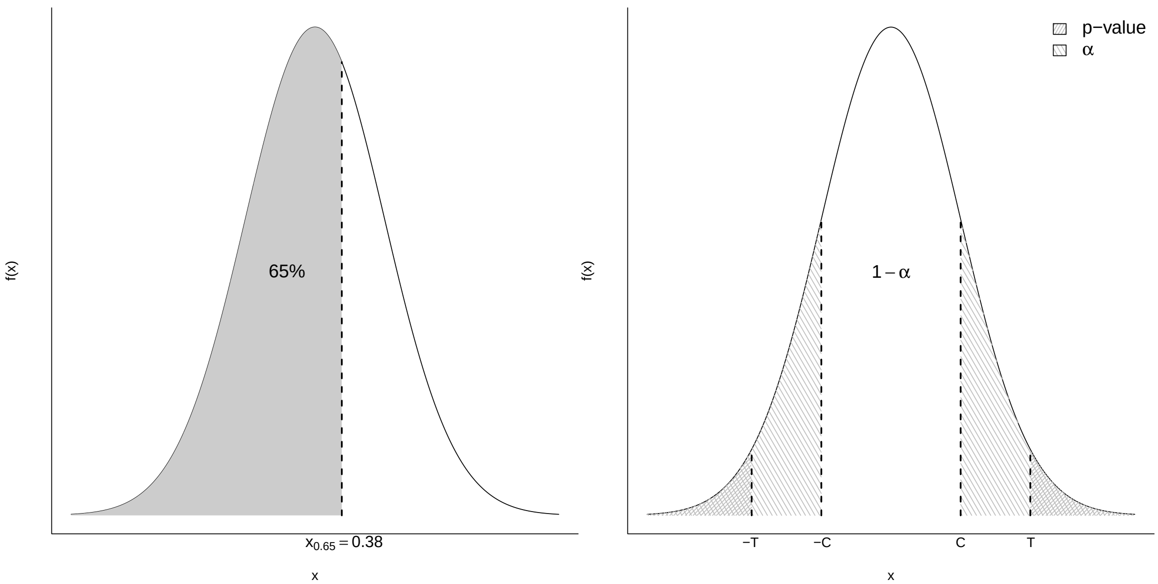
$$\mu = E(X) = \sum_{x} x f(x) = \frac{1}{n} \sum_{x} x. \tag{B.3}$$

Definition B.1.6. The *median* is the middle score for a set of data that has been arranged by order of magnitude. If there is an even number of observations, then the median is defined to be the average of the two middle values.[2] This measure is less affected by outliers and skewed data.

Definition B.1.7. The *variance* of a random variable X with mean $E(X)$ and the probability function $f(x)$, is:

$$\sigma^2 = Var(X) = E(X^2) - \mu^2. \tag{B.4}$$

[2] In this work, we take as median the minimum of the two middle values, since the average is not representative in a sample.



(a) (b)

Figure B.1: Remarkable values in a distribution function. (a) Representation of the 0.65 quantile. (b) Illustration of the critical value (C), confidence interval ([-C,C]), confidence level $(1-\alpha)$, significance level (α), and the test value (T) for determining the p-value (more extreme shaded values).

Definition B.1.8. The *standard deviation* of a random variable X is the positive square root of the variance of X, usually denoted by σ.

Definition B.1.9. Let X be a random variable. Then X is said to have the *normal distribution* if the distribution function of X is given by:

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}([y-\mu]/\sigma)^2} dy, \tag{B.5}$$

where μ and σ are the mean and standard deviation of X. The *standard normal distribution* is the normal distribution with μ equal to 0 and σ equal to 1.

Definition B.1.10. The number x_p for a given value of p between 0 and 1, is called the pth *quantile* of the random variable X, if $P(X < x_p) \leq p$ and $P(X > x_p) \leq 1-p$.

That is, X is less than x_p with probability p or less, and X exceeds x_p with probability $1 - p$ or less. The median is the 0.5 quantile, the third *decile* is the 0.3 quantile, the *upper* and *lower quartiles* are the 0.75 and 0.25 quantiles, respectively, and the sixty-five *percentile* is the 0.65 quantile.

The easiest method of finding the pth quantile involves using the graph of the distribution function of the random variable. The pth quantile is the abscissa of the point on the graph which has the ordinate value of p, as illustrated in Figure B.1-a.

Theorem B.1.1. (Central Limit Theorem) Let Y_n be the sum of n random variables $X_1, X_2, ..., X_n$, let μ_n be the mean of Y_n and let σ_n^2 be the variance of Y_n.

Table B.1: Measurement Levels of Data.

Level	Data Category	Description	Example
-	**Nominal/ categorical**	Numbers or labels are employed merely to identify mutually exclusive categories, but cannot be manipulated in a meaningful mathematical manner.	- "even" or "odd" - Indicator value
	Ordinal/ rank-order	The numbers represent rank-orders, and do not give any information regarding the differences between adjacent ranks.	- CPU benchmark - "HypE is better than MOEA/D"
	Interval	It considers the relative order of the measures involved but, in addition, the size of the interval between measurements.	- Temperature - "HypE is 60% better than MOEA/D "
+	**Ratio**	The same characteristics as in ordinal and interval, but also the ratio between two measurements is meaningful. It has a true zero point.	- "MOEA/D is twice faster than NSGA-II"

As n, the number of random variables, goes to infinity, the distribution function of the random variable

$$\frac{Y_n - \mu_n}{\sigma_n}$$

approaches the standard normal distribution function.

The theorem says that the distribution function of the sum of several random variables approaches the normal distribution function, as the number of random variables being added becomes large (i.e., goes to infinity), and when other general conditions are met [215].

We say that a *population*[3] is a collection of all elements under investigation, and a *sample* is a collection of some of these elements.

If each element in the population has an equal likelihood of being selected, then we refer to as a *random sample*. A more formal definition is the following:

Definition B.1.11. A *random sample* of size n is a sequence of n independent and identically distributed random variables $X_1, X_2, ..., X_n$.

On the other hand, the information is categorized with respect to the level of measurement that the data represents. Different levels of measurement contain different amounts of information, and hence, meaningful mathematical operations can be performed. In Table B.1, we present this classification.

Hypothesis testing is the process of inferring from a sample whether or not to accept a certain statement about the population. The statement itself is called the *hypothesis*. In each case the hypothesis is tested on the basis of the evidence contained in the sample. The hypothesis is either *rejected*, meaning the evidence from the sample

[3]This term should not be confused with the population concept in Evolutionary Computation, unless stated otherwise.

casts enough doubt on the hypothesis for us to say with some degree of confidence that the hypothesis is false, or *accepted*, meaning that it is not rejected.

The hypothesis to be tested is called the *null hypothesis* and is denoted by H_0. The *alternative hypothesis*, denoted by H_1, is the negation of the null hypothesis. The decision to reject H_0 is equivalent to the opinion "H_0 is false", and is equivalent to acceptance of H_1, or the opinion "H_1 is true". The decision to accept H_0 is not equivalent to the opinion "H_0 is true" but, instead, represents the opinion "H_0 has not been shown to be false", which could be the result of insufficient evidence. Therefore, if we wish to determine if a statement concerning some population is false, we make it the null hypothesis. If we wish to determine whether a statement is true, we make it the alternative hypothesis.

H_0 will often be of the form "samples A and B are drawn from the same distribution" or "samples A and B are drawn from distributions with the same mean value". Some other important definitions are presented next.

Definition B.1.12. A *test statistic* is a statistic used to help make the decision in a hypothesis test.

When a test is conducted and a claim is made about the hypotheses, two distinct errors are possible:

Definition B.1.13. The *type I error* is the action of rejecting H_0 when H_0 was actually true. The maximum probability of rejecting a true null hypothesis is usually labeled by α, and referred to as *significance level* of the test.

Definition B.1.14. The *type II error* is an action of failing to reject H_0 when H_1 was actually true. The probability of the type II error is denoted by β. *Power* is defined as $1-\beta$. In simple terms, the power is propensity of a test to reject wrong alternative hypothesis.

Definition B.1.15. The *critical region* is the set of all points in the sample space that result in the decision to reject the null hypothesis.

Definition B.1.16. The probability, assuming H_0 is true, that the test statistic would take a value as extreme or more extreme than actually observed is called the *p-value.*

The main difference between α and p-value is that α is a parameter defined by the user, while the p-value is the result of a statistical test applied to the sampled data. The smaller the α, the more stringent the test. The smaller the p-value, the stronger the evidence is in favor of the alternative hypothesis.

To determine if an observed outcome is statistically significant, we compare the values of α and the p-value. There are two possibilities that emerge:

- If the p-value is less than or equal to the α, then the null hypothesis should be rejected, and the result is said to be statistically significant.

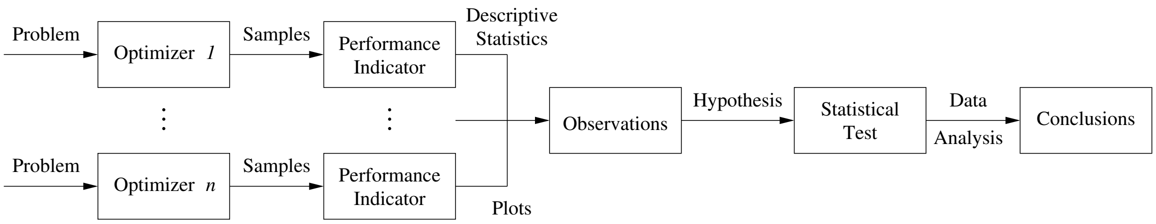


Figure B.2: Flow chart of the experimental methodology.

- If the p-value is greater than α, then we fail to reject the null hypothesis, and we say that the result is statistically non-significant.

Definition B.1.17. A $1-\alpha$ level *confidence interval* is a statistic, in the form of a region or interval, that contains an unknown parameter with probability $1-\alpha$.

The confidence interval can take any number of probabilities, with the most common being 95% or 99%. Some of these definitions are depicted in Figure B.1-b.

B.2 Experimental Methodology

In this section we describe the statistical comparison methodology, following the recommendations made by Knowles et al. [216]. It is worth noting that these steps rely on the scientific method [217].

One of the major goals in Evolutionary Computation is to check whether an algorithm provides significantly better approximation sets than another optimizer with respect to some test problem. To determine this fact, random samples of finite size must be generated by each optimizer, and then, every approximation set in objective space is transformed into a real value using a performance indicator that should be compliant with Pareto dominance. Next, the resulting data is summarized using descriptive statistics, such as the mean and variance,[4] in order to build tables and box-plots [218] that are useful to identify patterns, make observations, and in consequence, ask some question that will be formulated in a hypothesis. Such hypothesis will be accepted or rejected by means of an statistical test. From these results, we are now ready to draw conclusions. This process is depicted in Figure B.2.

The use of a quality indicator reduces the dimension of an approximation set to a single figure of merit. In many studies on multi-objective algorithm performance, more than one Pareto compliant indicator is used to compare approximation sets. If that is the case, the data on which the test has been carried out should be used once, and in consequence, a new independent data must be generated to make more inferences. In the case when the tests are computationally expensive, it is possible

[4] The mean, median and mode are sometimes referred to as first order moments of a distribution or measures of central tendency, and they describe or summarize the location of the distribution on the real number line. The variance, standard deviation, and inter-quartile range are known as second-order moments or measures of variability and they describe the spread of the data.

to use methods for correcting the p-values for the reduction in confidence, like the Bonferroni correction [216] or methods based on re-samplings [219].

Some statistical tests are based on assuming the data is drawn from a distribution that closely approximates a known distribution, for example the normal distribution. Such known distributions are completely defined by their parameters (e.g. the mean and standard deviation), and tests based on these known distributions are thus termed parametric statistical tests. However, the assumption of normality cannot be theoretically justified for stochastic optimizers, in general, and it is difficult to empirically test for normality with relatively small samples (less than 100 runs). Therefore, it is safer to rely on nonparametric tests, which make no assumptions about the distributions of the variables.

There exist two main types of nonparametric tests: rank tests and permutation tests. Rank tests pool the values from several samples and convert them into ranks by sorting them, and then employ tables describing the limited number of ways in which ranks can be distributed (between two or more algorithms) to determine the probability that the samples come from the same source. Permutation tests use the original values without converting them to ranks but estimate the likelihood that samples come from the same source explicitly by Monte Carlo simulation [220].

Rank tests are the less powerful but are also less sensitive to outliers and are computationally cheap. If there are just two optimizers, for example, the Wilcoxon rank sum test [215, 221] can be applied. The Kruskal-Wallis rank test [215, 221] is an extension that works for multiple algorithms. Permutation tests are more powerful because information is not thrown away. They are also better when there are many tied values in the samples and, in certain circumstances, when it may be important to compare the worst-case or best-case performance of optimizers. However, they can be expensive to compute for large samples. Examples of permutation methods are Bootstrapping [221, 222], Jacknife [221] or Fisher's permutation test [215, 221].

When comparing a pair of stochastic optimizers, two slightly different scenarios are possible. In one case, each run of each optimizer is a completely independent random sample; that is, the initial population,[5] the random seed, and all other random variables are drawn independently at random on each run. In the other case, the influence of one or more random variables is partially removed from consideration; i.e., the initial population used by the two algorithms may be matched in corresponding runs, so that the runs (and hence the final quality indicator values) should be taken as pairs. In the former scenario, the statistical testing will reveal, in quite general terms, whether there is a difference in the distributions of indicator values resulting from the two stochastic optimizers, from which a general performance difference can be inferred. In the latter scenario, the statistical testing reveals whether there is a difference in the indicator value distributions given the same initial population, and the inference in this case relates to the optimizer's ability to improve the initial population. While the former scenario is more general, the latter may give more statistically significant results. If matched samples have been collected, then the

[5] Here, we refer to the term population in Evolutionary Computation

Table B.2: Decision Table for Inferential Statistical Test. The measurement scale is indicated by the symbol + (at least ordinal) or - (at least interval).

<table>
<tr><th>Number of Random Samples</th><th>Independent Samples</th><th>Dependent Samples</th><th>Hypothesis Test Involving</th></tr>
<tr><td>Two</td><td>+ Wilcoxon rank sum test (a, b, d, e)
- Squared ranks test (c, d)
+ Klotz test (c, d)
+ Kolmogorov-Smirnov test (b, d, e)
+ Cramér-von Mises test (d)
- Randomization test or Fisher's permutation test (a)
+ Bootstrap (a, c, d)
+ Jacknife (a, c, d)
+ Siegel-Tukey test (c)
- Moses test (c)</td><td>- Wilcoxon matched-pairs signed-ranks test (a, b, d, e)

+ Binomial sign test (b, d, e)

- Randomization test for matched pairs or Fisher's matched samples test (a)</td><td rowspan="2">a. Means (medians)

b. Confidence interval

c. Variances

d. Identical populations or distributions

e. Ordering</td></tr>
<tr><td>Several</td><td>+ Median test (a)
+ Kruskal-Wallis test (a, d, e)
+ van der Waerden test (a, d, e)
- Squared ranks test (c, d)
+ Birnbaum-Hall test (d)
+ k-sample Smirnov test (d, e)</td><td>+ Friedman two-way analysis of variance by ranks (a)</td></tr>
</table>

Wilcoxon matched-pairs signed-ranks test or Fisher's matched samples test [215] can be used. Similarly, the Friedman test [221] extends to any number of related samples.

In Table B.2, we provide a classification of the most popular statistical tests [215, 221, 223], separated by the number of samples to be evaluated, dependence, measurement scale, and the property they evaluate; such as if the hypothesis is based on the mean/median/variance, or if the distributions are similar, or if the test provides some indication about the ordering of distributions.

There are several statistical software packages that implement these tests. Here is a short list of them [224]:

Matlab `http://www.mathworks.com`

R-project `www.r-project.org`

StatXact `http://www.cytel.com/software/statxact`

Minitab `http://www.minitab.com`

TESTIMATE `http://www.idv-cro.eu/cms/index.php?id=testimate&L=1`

statgraphics `http://www.statgraphics.com`

SPSS `http://www-01.ibm.com/software/analytics/spss`

BMDP `http://www.statistical-solutions-software.com`

SAS `http://www.sas.com`

Stata `http://www.stata.com`

STATISTICA `http://www.statsoft.com`

In the following section, we will describe the Wilcoxon rank sum test, which was used to deduce the inferences in Chapter 5.

B.3 The Wilcoxon Rank Sum Test

This test, also known as the Mann-Whitney U test, was invented by Frank Wilcoxon[6] in 1945. Later, Mann and Whitney (1947) considered unequal sample sizes and furnished tables suitable for use with small samples. Equivalent forms of the same test appeared in the literature under other names (see [215, 224]). This nonparametric test is employed with ordinal data in a hypothesis testing situation involving a design with two independent samples. If the result is significant, it indicates there is a significant difference between the two sample medians, and as a result of the latter, the researcher can conclude that there is a high likelihood that the samples represent populations with different median values.

In this case, ranks may be considered preferable to the actual data, since they retain only the ordering of the observations and make no other use of their numerical values. Additionally, the probability theory of statistics based on ranks is relatively simple and does not make any specific assumptions about the shape of the distribution, such as normality. Another reason for preferring ranks is that outliers are eliminated, thus the influence in variability is dramatically reduced. However, some information is sacrificed in the transformation of the data from interval/ratio into ranks.

Assumptions

The Wilcoxon rank sum test is based on the following assumptions:

1. Each sample has been randomly selected from the population it represents.

2. In addition to independence within each sample, there is mutual independence between the two samples.

3. The original variable observed (which is subsequently ranked) is a continuous random variable.

4. The underlying distributions from which the samples are derived are identical in shape.

[6]Wilcoxon (1892-1965) was a chemist who encountered statistical problems in his work at the research laboratories of the American Cyanimid Company.

Hypotheses

Let $E(X)$ and $E(Y)$ be the means of the populations X and Y, respectively. Then, the hypotheses may be stated as follows:

A. Two-Tailed Test
$H_0 : E(X) = E(Y)$ versus $H_1 : E(X) \neq E(Y)$

B. One-Tailed Test
$H_0 : E(X) \geq E(Y)$ versus $H_1 : E(X) < E(Y)$

C. One-Tailed Test
$H_0 : E(X) \leq E(Y)$ versus $H_1 : E(X) > E(Y)$

The definition of the alternative hypothesis usually takes one of two forms. If H_1 is of the form "sample X comes from a better distribution than sample Y", then the inferential test is a one-tailed test. If H_1 does not specify a prediction about which distribution is better, and is of the form "sample X and sample Y are from different distributions", then it is a two-tailed test. A one-tailed test is more powerful than a two-tailed test, meaning that for a given α value, it rejects the null hypothesis more readily in cases where it is actually false.

In Algorithm 20, we present the Wilcoxon rank sum test [215]. In line 1, the ranking assignment is performed. In this case, both samples are combined into a single ordered sample, and then, the ranks are assigned to each observation, starting from 1, for the smallest value, to $n + m$, for the largest, without regard to the population each value came from. If several sample values are exactly equal to each other (tied), it is assigned to each the average of the ranks that would have been assigned to them if there had been no ties. Then, in line 2, the statistic W is calculated. If there are no ties or just a few, it is equal to the sum of the ranks assigned to the values from the sample X. Otherwise, it is the subtracted mean from the previous value, and divided by the standard deviation. In the following lines, the decision rule is applied depending on whether the hypothesis of interest is classified as A, B, or C. Here, w_p is the pth quantile of the random variable W (see Definition B.1.10). If W was calculated using the formula for many ties, then the quantile can be obtained directly from the quantiles of the standard normal distribution. Otherwise, the upper quantiles may be computed by subtraction from $n(n + m + 1)$. That is,

$$w_{1-p} = n(n + m + 1) - w_p. \tag{B.6}$$

As an alternative to using upper quantiles, the statistic W', defined as:

$$W' = n(n + m + 1) - W, \tag{B.7}$$

may be used; just the flow of decision rules changes in Algorithm 20, since instead of rejecting the null hypothesis, this is accepted, and vice versa.

Algorithm 20 Wilcoxon Rank Sum Test

Input: The random samples $X_1, X_2, ..., X_n$ and $Y_1, Y_2, ..., Y_m$ from two distinct populations, the significance level α and the hypothesis *type*.

Output: Rejection or acceptance of H_0 at the level of significance α.

Assign the ranks $R(X_i)$ and $R(Y_j)$ for all $i \in \{1, ..., n\}$ and $j \in \{1, ..., m\}$

Compute the Wilcoxon rank sum statistic:

$$W \leftarrow \begin{cases} \sum_{i=1}^{n} R(X_i) & \text{if there are few ties} \\ \dfrac{\sum_{i=1}^{n} R(X_i) - n\frac{N+1}{2}}{\sqrt{\frac{nm}{N(N-1)}\left(\sum_{i=1}^{n} R(X_i)^2 + \sum_{j=1}^{m} R(Y_j)^2\right) - \frac{nm(N+1)^2}{4(N-1)}}} & \text{otherwise,} \end{cases}$$

where $N = n + m$.

if *type* is A **then**
  Calculate the quantiles $w_{\alpha/2}$ and $w_{1-\alpha/2}$
  if $W < w_{\alpha/2}$ **or** $W > w_{1-\alpha/2}$ **then**
    Reject H_0
  else
    Accept H_0
else if *type* is B **then**
  Calculate the quantile w_α
  if $W < w_\alpha$ **then**
    Reject H_0
  else
    Accept H_0
else if *type* is C **then**
  Calculate the quantile $w_{1-\alpha}$
  if $W > w_{1-\alpha}$ **then**
    Reject H_0
  else
    Accept H_0

Because W is the sum of the ranks of the nXs, for large n and m the central limit theorem (see Theorem B.1.1) may be applied to obtain an approximate distribution for W. Therefore, W is approximately normal with mean and variance:

$$E(W) = \frac{n(n+m+1)}{2}, \tag{B.8}$$

and

$$Var(W) = \frac{nm(n+m+1)}{12}. \tag{B.9}$$

Thus, the quantiles of W may be approximated as:

$$w_p = E(W) + x_p\sqrt{Var(W)}, \tag{B.10}$$

where x_p is the pth quantile of the standard normal distribution.

It should be noted that if the sum is too small (or too large) with respect to its mean, there is some indication that the values from that population tend to be smaller (or larger, as the case may be) than the values from the other population. Hence, for the case A, the null hypothesis of no differences between populations may be rejected if the ranks associated with one sample tend to be larger than those of the other sample. For case B, small values of W indicate that H_1 is true, and in consequence for C, large values of W indicate that H_1 is true.

Example

As an illustrative example, suppose we want to evaluate the performance of two optimizers on the DTLZ1 test problem for two objectives,[7] so each algorithm is executed independently at random 10 times,[8] and from the outcome Pareto front approximations, the hypervolume is computed using the reference point at $(1, 1)$. The adopted significance level is $\alpha = 0.05$.

The indicator values of *Optimizer 1* (X_i) and *Optimizer 2* (Y_i) are shown in the second column of Table B.3. According to Figure B.3, one may guess that *Optimizer 1* outperforms *Optimizer 2* (assuming the hypervolume is to be maximized). Therefore, the Wilcoxon rank sum test can be applied to verify this fact. Since we wish to determine if the assumption is true, we make it in H_1. Thus, the hypotheses are:

- H_0: *Optimizer 1* does not perform better on DTLZ1 than *Optimizer 2*.
- H_1: *Optimizer 1* performs better on DTLZ1 than *Optimizer 2*.

The null hypothesis could also be stated as $H_0 : E(X) \leq E(Y)$ versus $H_1 : E(X) > E(Y)$, according to the set C of hypotheses.

Following the steps of Algorithm 20, the values are ranked, as it is shown in the third column of Table B.3. Since there are seven groups of tied values, the ranks are averaged within each group, as it is shown in the fourth column.

The next step is to calculate the Wilcoxon rank sum statistic for many ties. Then, we have $n = m = 10$, so $N = 20$. Then, the sum of the ranks assigned to Xs is:

$$\begin{aligned} W' &= \sum_{i=1}^{n} R(X_i) \\ &= 2 + 4.5 + 12.5 + 14 + 15 + (3)(17) + 19 + 20 \\ &= 138, \end{aligned}$$

[7]See Appendix A for the problem definition.

[8]The parameters employed are the same as the described in Chapter 5.

Table B.3: Indicator values and rank assignment for the example.

Optimizer	Hypervolume	Tied Rank	Final Rank
1	0.87351	1	2
2	0.87351	2	2
2	0.87351	3	2
1	0.87352	4	4.5
2	0.87352	5	4.5
2	0.87353	6	6.5
2	0.87353	7	6.5
2	0.87354	8	8.5
2	0.87354	9	8.5
2	0.87355	10	10.5
2	0.87355	11	10.5
1	0.87356	12	12.5
2	0.87356	13	12.5
1	0.87358	14	14
1	0.87359	15	15
1	0.87361	16	17
1	0.87361	17	17
1	0.87361	18	17
1	0.87362	19	19
1	0.87363	20	20

and the sum of the squares of all 20 ranks is:

$$\sum_{i=1}^{n} R(X_i)^2 + \sum_{j=1}^{m} R(Y_j)^2 = 2863.5.$$

Now, we can compute W:

$$\begin{aligned} W &= \frac{W' - n\frac{N+1}{2}}{\sqrt{\frac{nm}{N(N-1)}\left(\sum_{i=1}^{n} R(X_i)^2 + \sum_{j=1}^{m} R(Y_j)^2\right) - \frac{nm(N+1)^2}{4(N-1)}}} \\ &= \frac{138 - 10\frac{20+1}{2}}{\sqrt{\frac{(10)(10)}{20(20-1)}(2863.5) - \frac{(10)(10)(20+1)^2}{4(20-1)}}} \\ &= 2.5068. \end{aligned}$$

In the following, we calculate the $w_{1-\alpha}$ quantile, which is equivalent to the $x_{1-\alpha}$ quantile of the standard normal distribution. Thus, assuming $w_{0.95} = 1.6449$,[9] we

[9]This value was obtained directly from tables (see Table A1 from [215]). It can also be approximated using statistical software packages, as we will see later in this Appendix.

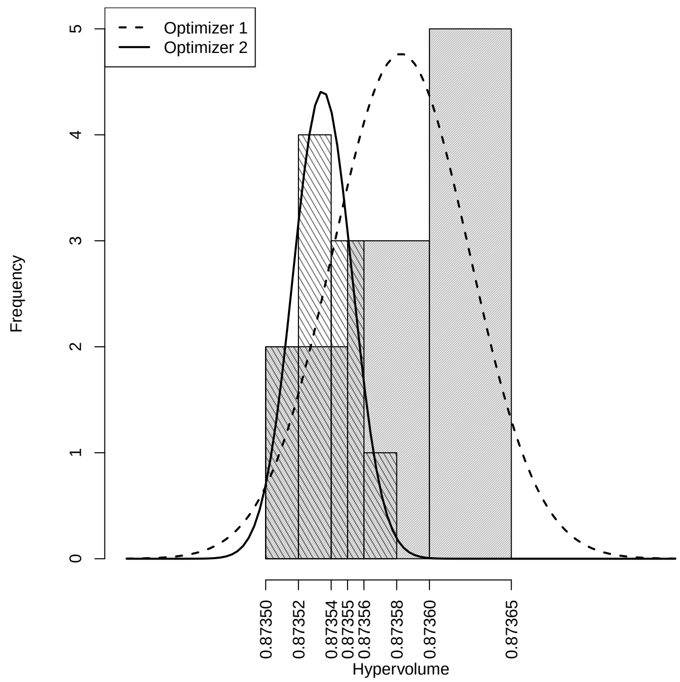


Figure B.3: Histogram and normal approximation curve for the example.

notice that $W > w_{0.95}$, i.e., $2.5068 > 1.6449$; therefore. the null hypothesis is rejected at the significance level of 0.05.

On the other hand, if we had used the Wilcoxon rank sum statistic for few ties, $W' = 138$, then we would have obtained from equations (B.8), (B.9), (B.10), and assuming $x_{0.05} = -1.6449$:

$$E(W) = \frac{n(n+m+1)}{2} = \frac{10(20+1)}{2} = 105.$$

$$Var(W) = \frac{nm(n+m+1)}{12} = \frac{(10)(10)(20+1)}{12} = 175.$$

$$\begin{aligned} w_{0.05} &= E(W) + x_{0.05}\sqrt{Var(W)} \\ &= 105 - 1.6449\sqrt{175} \\ &= 83.24, \end{aligned}$$

and finally, from eq. (B.6):

$$\begin{aligned} w_{0.95} &= n(n+m+1) - w_p \\ &= 10(20+1) - 83.24 \\ &= 126.7599, \end{aligned}$$

which is less than W', i.e., 138 > 126.7599, and again, H_0 is rejected.

From these results, we can conclude that:

- "*Optimizer 1* performs significantly better on DTLZ1 than *Optimizer 2* with respect to the hypervolume indicator, using a significance level α of 5%."

Another way to express this idea is:

- "We are 95% confident that *Optimizer 1* performs significantly better on DTLZ1 than *Optimizer 2* with respect to the hypervolume indicator."

As we mentioned before, the Wilcoxon rank sum test can be obtained from several software packages. Here, we present an implementation of the previous example in Matlab and R-project.

The function *ranksum* of Matlab performs a two-sided rank sum test of the hypothesis that two independent samples come from distributions with equal medians. It returns the following information:

- p-value
- h, which indicates if H_0 can be rejected ($h = 1$) or not ($h = 0$) at the significance level α.
- $stats$, which is composed of $zval$ (the value of the normal statistic) and $ranksum$ (statistic W).

The following code computes this test for the previous example, assuming the files *optimizer1.dat* and o*ptimizer2.dat* contain the hypervolume values of the two optimizers.

```
fid = fopen('optimizer1.dat');
a = fscanf(fid, '%g', [1 inf]);
fclose(fid);

fid = fopen('optimizer2.dat');
b = fscanf(fid, '%g', [1 inf]);
fclose(fid);

[p-value,h,stats] = ranksum(a,b)
```

Output:

```
p-value = 0.0136

h = 1

stats =
       zval: 2.4689
    ranksum: 138
```

Similarly, the function *wilcox.test* of R-project performs the one and two-sided tests. It is worth mentioning that when there are tied values, an approximation of the p-value is computed (see `help(wilcox.test)` for more details).

The implementation of the previous example is given in the following lines:

```
x = read.table(file="optimizer1.dat",header=FALSE)$V1
y = read.table(file="optimizer2.dat",header=FALSE)$V1

wilcox.test(x,y, alternative="greater",conf.level=0.95,exact=FALSE)
```

Output:

```
        Wilcoxon rank sum test with continuity correction

data:  x and y
W = 83, p-value = 0.006777

alternative hypothesis: true location shift is greater than 0
```

Since in both cases the p-value is less than the 0.05 significance level, the null hypothesis is again rejected.

B.4 Summary

This appendix has presented a rigorous statistical methodology for the comparison of MOEAs, which has strong links with the scientific method.

Even though when looking at the data or graph of one optimizer it may seem better than another, it is necessary to validate its performance in a statistical way. The use of descriptive statistics is limited, and should usually be given only to supplement any statistical inferences that can be made from the data.

The most commonly used methods for inference assume that the variables in question have a Normal distribution in the populations. In practice, this is, of course, false, and if after inspection using plots or applying some test of normality,[10] the data is clearly not Normal, we can adopt some of the following approaches:

- Nonparametric methods: These tests use ranks instead of actual values. Examples are Wilcoxon test, the Kolmogorov-Smirnov test, Fisher's test, etc.

- Permutation methods: These methods are computationally expensive and must be applied for representative samples (not small). Examples of this kind of test are the Bootstrap and Jacknife tests.

[10] For example: D'Agostino's K-squared test, the Jarque–Bera test, the Anderson–Darling test, the Cramér–von Mises criterion or the Lilliefors test (see [215]).

When we are interested in the performance comparison of algorithms, we need to be confident in the conclusion. Even if the average is better, we need to take variance into account, so that we can assess our results in a more appropriate way. Sometimes, we would need to find the confidence intervals around both means and verify that they do not overlap. In some other occasions, we would want to test if one algorithm is better than its previous version. For this scenario, we need to check if the difference between means is greater than zero. Another recommendation is to minimize the number of different tests carried out on the same data by carefully choosing which tests to apply before collecting the data. That is, we must not do tests unless there is some realistic chance that the null hypothesis can be rejected, and in consequence, the result would be interesting. It is also not advisable to reuse data in more than one inference.

Additionally, it is important to mention, that one should not claim that MOEAs are the only algorithms able to solve a set of test problems efficiently and effectively, since the *No Free Lunch Theorem* (see the last part of Section 2.6 in Chapter 2) indicates that stochastic algorithms are not individually robust over all problems, by definition.

Appendix C

Numerical Results of Experiments

This appendix presents a summary of the numerical results obtained from the experiments made in Chapter 5.[1] The data is organized in tables and figures as follows:

- **Comparison table of optimizers:** The values of a performance indicator for 100 independent runs from 2 to 10 objective functions are summarized using descriptive statistics, such as the median (med.), mean or average (avg.), minimum (min.), maximum (max.) and standard deviation (std.). The three best values are shown in gray scale, where the darker tone corresponds to the best value. In the case of the hypervolume indicator, a zero value means that the approximation set in objective space is outside the enclosed region described by the reference point. This indicates that, the optimizer did not converge to the Pareto optimal front.

- **Box-plot of optimizers:** The information of the previous point is represented graphically using box-plots (outliers are omitted). Optimizers are sorted from the best to the worst median for each dimension. In the case of the hypervolume indicator, the widths of the boxes are proportional to the number of samples different from zero.

- **Statistical test table:** The Wilcoxon rank sum test (one-tailed) is applied to hypervolume indicator values in order to compare the performance of approximation sets. The information shown in these tables corresponds to the p-values computed in the R-project. A p-value greater than 0.05 (the significance level α) means that there is not enough evidence that one optimizer performs better than another one in a specific problem of certain dimensionality. No information is shown if the hypervolume indicator value is zero.

- **Approximation plot:** The Pareto optimal front (POF) and True Pareto front (True PF) is depicted for 2D and 3D. For higher dimensionality, parallel coordinates are used for representing approximation sets.

[1]Due to its extent, the complete version is included on the CD that accompanies this report or is available for download at: `http://computacion.cs.cinvestav.mx/~rhernandez/mombi`.

Table C.1: Comparison of hypervolume indicator values for different optimizers on the DTLZ1 test problem.

Dim.	Stat.	NSGA-II	MOEA/D TCH	MOEA/D NTCH	MOEA/D PBI	SMS-EMOA	Δ_p-DDE	R2-MOGA	R2-MOGAw	R2-MODE	R2-IBEA	MOMBI TCH	MOMBI NTCH	MOMBI PBI
2D	med.	8.734346e-01	8.738003e-01	8.737079e-01	8.735956e-01	8.738857e-01	8.724069e-01	8.673199e-01	8.691769e-01	8.716020e-01	8.737985e-01	8.738110e-01	8.733910e-01	8.736873e-01
	avg.	8.733844e-01	8.737439e-01	8.732419e-01	8.735514e-01	8.738489e-01	8.722998e-01	8.670841e-01	8.686361e-01	8.704983e-01	8.737835e-01	8.737656e-01	8.733319e-01	8.736318e-01
	min.	8.722344e-01	8.713626e-01	8.531973e-01	8.725104e-01	8.733583e-01	8.704749e-01	8.574558e-01	8.563981e-01	8.618752e-01	8.734285e-01	8.731185e-01	8.720716e-01	8.729240e-01
	max.	8.737079e-01	8.739366e-01	8.739218e-01	8.739280e-01	8.739452e-01	8.739266e-01	8.714175e-01	8.721508e-01	8.727579e-01	8.738595e-01	8.739348e-01	8.735767e-01	8.738774e-01
	std.	2.440076e-04	2.748538e-04	2.206064e-03	2.654077e-04	1.068987e-04	6.338654e-04	2.603557e-03	2.998287e-03	2.662856e-03	7.160696e-05	1.549755e-04	2.136591e-04	2.153411e-04
3D	med.	9.555524e-01	9.689333e-01	8.708503e-01	9.742206e-01	9.745157e-01	9.637425e-01	9.641550e-01	9.621279e-01	9.704498e-01	9.742265e-01	9.689257e-01	9.690623e-01	9.742623e-01
	avg.	9.550525e-01	9.689145e-01	8.704594e-01	9.741546e-01	9.744879e-01	9.635953e-01	9.638756e-01	9.616669e-01	9.703763e-01	9.742197e-01	9.689603e-01	9.689667e-01	9.742264e-01
	min.	9.249775e-01	9.681165e-01	8.009318e-01	9.734604e-01	9.741681e-01	9.425283e-01	9.577672e-01	9.498465e-01	9.674370e-01	9.740395e-01	9.683805e-01	9.649935e-01	9.735498e-01
	max.	9.687004e-01	9.693121e-01	9.630539e-01	9.744246e-01	9.745695e-01	9.730126e-01	9.682085e-01	9.680136e-01	9.717142e-01	9.743164e-01	9.699062e-01	9.707590e-01	9.744226e-01
	std.	8.591776e-03	1.763741e-04	2.989390e-02	2.153037e-04	7.440985e-05	6.836488e-03	2.520996e-03	3.748257e-03	7.861322e-04	5.325901e-05	2.587439e-04	1.042212e-03	1.650704e-04
4D	med.	9.639580e-01	9.884687e-01	8.624844e-01	9.943298e-01	8.047787e-01	9.747133e-01	9.864824e-01	9.850237e-01	9.925731e-01	9.944370e-01	9.884326e-01	9.873143e-01	9.943974e-01
	avg.	8.154121e-01	9.884614e-01	8.480667e-01	9.943105e-01	8.654303e-01	9.711843e-01	9.862139e-01	9.847283e-01	9.925270e-01	9.944316e-01	9.884203e-01	9.865345e-01	9.943703e-01
	min.	1.837876e-02	9.882232e-01	3.228452e-01	9.938169e-01	7.719201e-01	9.110759e-01	9.808208e-01	9.773680e-01	9.913645e-01	9.943297e-01	9.880964e-01	9.794132e-01	9.937398e-01
	max.	9.854703e-01	9.886414e-01	9.405262e-01	9.944755e-01	9.936205e-01	9.923599e-01	9.897951e-01	9.903498e-01	9.930385e-01	9.944927e-01	9.888881e-01	9.895632e-01	9.944775e-01
	std.	2.639458e-01	7.778014e-05	6.207241e-02	1.268563e-04	9.245684e-02	1.570521e-02	1.764767e-03	2.631145e-03	3.039182e-04	2.980228e-05	1.412847e-04	2.451571e-03	1.050998e-04
5D	med.	0.000000e+00	9.967163e-01	8.537713e-01	9.985024e-01	9.918092e-01	9.814189e-01	9.935068e-01	9.933993e-01	9.980028e-01	9.987106e-01	9.967983e-01	9.939385e-01	9.986310e-01
	avg.	0.000000e+00	9.962519e-01	8.343190e-01	9.984717e-01	9.589651e-01	9.776263e-01	9.933586e-01	9.931676e-01	9.980065e-01	9.987085e-01	9.967950e-01	9.935337e-01	9.986193e-01
	min.	0.000000e+00	9.905500e-01	2.543076e-01	9.977365e-01	3.506129e-01	9.314181e-01	9.895773e-01	9.888739e-01	9.976485e-01	9.986677e-01	9.966289e-01	9.870903e-01	9.983420e-01
	max.	0.000000e+00	9.968710e-01	9.305960e-01	9.986334e-01	9.983578e-01	9.961791e-01	9.955494e-01	9.962413e-01	9.982529e-01	9.987327e-01	9.968840e-01	9.961862e-01	9.986928e-01
	std.	0.000000e+00	1.414327e-03	7.842749e-02	1.364560e-04	9.349789e-02	1.359742e-02	1.176658e-03	1.802964e-03	1.051194e-04	1.518398e-05	4.314019e-05	1.797592e-03	4.847156e-05
6D	med.	0.000000e+00	9.951489e-01	8.463574e-01	9.994771e-01	9.920169e-01	9.575644e-01	9.961916e-01	9.962213e-01	9.994027e-01	9.996664e-01	9.978145e-01	9.946612e-01	9.995881e-01
	avg.	0.000000e+00	9.932675e-01	8.211537e-01	9.994641e-01	9.380970e-01	9.560991e-01	9.959681e-01	9.959699e-01	9.994011e-01	9.996662e-01	9.977215e-01	9.944595e-01	9.995843e-01
	min.	0.000000e+00	9.616526e-01	1.221068e-01	9.991791e-01	9.921699e-02	9.080335e-01	9.919554e-01	9.899476e-01	9.992723e-01	9.996433e-01	9.963297e-01	9.821905e-01	9.994191e-01
	max.	0.000000e+00	9.980609e-01	9.330311e-01	9.996062e-01	9.995476e-01	9.929916e-01	9.981697e-01	9.979415e-01	9.995203e-01	9.996791e-01	9.986019e-01	9.980946e-01	9.996362e-01
	std.	0.000000e+00	5.651639e-03	9.257611e-02	7.989127e-05	1.330509e-01	1.823559e-02	1.109005e-03	1.308754e-03	4.010875e-05	6.978851e-06	6.360691e-04	2.108687e-03	3.396471e-05
7D	med.	0.000000e+00	9.872554e-01	8.315116e-01	9.997551e-01	9.519868e-01	9.546691e-01	9.944006e-01	9.946647e-01	9.996740e-01	9.998409e-01	9.927873e-01	9.811349e-01	9.998043e-01
	avg.	0.000000e+00	9.816736e-01	8.221327e-01	9.997416e-01	9.099563e-01	8.609945e-01	9.940195e-01	9.946996e-01	9.996599e-01	9.998402e-01	9.925764e-01	9.797401e-01	9.997975e-01
	min.	0.000000e+00	9.515551e-01	7.579787e-01	9.995612e-01	2.364616e-01	1.921871e-01	9.892251e-01	9.891611e-01	9.994752e-01	9.998041e-01	9.857445e-01	9.628000e-01	9.997339e-01
	max.	0.000000e+00	9.952074e-01	9.442175e-01	9.998397e-01	9.995339e-01	9.973798e-01	9.973009e-01	9.977685e-01	9.997396e-01	9.998633e-01	9.956641e-01	9.916458e-01	9.998370e-01
	std.	0.000000e+00	1.123620e-02	3.208858e-02	5.498313e-05	1.149606e-01	1.963424e-01	1.802111e-03	1.821607e-03	5.357032e-05	1.165437e-05	1.806478e-03	5.835605e-03	2.501178e-05
8D	med.	0.000000e+00	9.803649e-01	8.314924e-01	9.995553e-01	9.853202e-01	5.987649e-01	9.972330e-01	9.975402e-01	9.999318e-01	9.999584e-01	9.938804e-01	9.853580e-01	9.998455e-01
	avg.	0.000000e+00	9.765100e-01	8.061287e-01	9.995014e-01	8.895187e-01	6.857738e-01	9.971573e-01	9.972795e-01	9.999307e-01	9.999567e-01	9.936584e-01	9.854466e-01	9.998382e-01
	min.	0.000000e+00	9.496235e-01	5.581130e-02	9.986764e-01	1.550827e-02	5.967515e-01	9.945028e-01	9.928544e-01	9.999002e-01	9.999373e-01	9.872191e-01	9.780277e-01	9.996004e-01
	max.	0.000000e+00	9.957381e-01	8.963430e-01	9.998332e-01	9.998264e-01	9.972331e-01	9.986403e-01	9.991893e-01	9.999464e-01	9.999653e-01	9.962274e-01	9.938527e-01	9.999354e-01
	std.	0.000000e+00	1.297069e-02	1.051993e-01	2.273272e-04	1.917848e-01	1.439005e-01	8.544291e-04	1.072928e-03	7.180052e-06	6.339493e-06	1.707611e-03	3.118081e-03	5.384674e-05
9D	med.	0.000000e+00	9.777119e-01	8.332023e-01	9.971328e-01	9.508728e-01	6.016146e-01	9.986087e-01	9.987843e-01	9.999860e-01	9.999879e-01	9.942508e-01	9.861928e-01	9.996873e-01
	avg.	0.000000e+00	9.760705e-01	8.161496e-01	9.967018e-01	7.548349e-01	6.473564e-01	9.985447e-01	9.986536e-01	9.999859e-01	9.999874e-01	9.940920e-01	9.867293e-01	9.996597e-01
	min.	0.000000e+00	9.472606e-01	5.570900e-02	9.906709e-01	9.819143e-03	2.282549e-01	9.965181e-01	9.965254e-01	9.999807e-01	9.999763e-01	9.882504e-01	9.797701e-01	9.992602e-01
	max.	0.000000e+00	9.943222e-01	9.381262e-01	9.986328e-01	9.994021e-01	9.927075e-01	9.994551e-01	9.993544e-01	9.999886e-01	9.999919e-01	9.971039e-01	9.931990e-01	9.998756e-01
	std.	0.000000e+00	1.142592e-02	8.415794e-02	1.447313e-03	3.302179e-01	1.085995e-01	4.764604e-04	5.504621e-04	1.376429e-06	2.715327e-06	1.663979e-03	2.424245e-03	1.397253e-04
10D	med.	0.000000e+00	9.690008e-01	8.327973e-01	9.899974e-01	9.693538e-01	7.087199e-01	9.991662e-01	9.991788e-01	9.999971e-01	9.999944e-01	9.948660e-01	9.868172e-01	9.993001e-01
	avg.	0.000000e+00	9.694891e-01	8.124819e-01	9.889401e-01	8.809190e-01	7.087199e-01	9.990837e-01	9.991686e-01	9.999971e-01	9.999939e-01	9.937261e-01	9.874692e-01	9.992103e-01
	min.	0.000000e+00	9.022552e-01	6.700127e-02	9.665092e-01	2.171787e-01	7.087199e-01	9.974336e-01	9.979344e-01	9.999966e-01	9.999857e-01	8.883990e-01	9.841380e-01	9.976498e-01
	max.	0.000000e+00	9.934270e-01	9.161362e-01	9.947932e-01	9.999511e-01	7.087199e-01	9.997017e-01	9.997776e-01	9.999976e-01	9.999968e-01	9.976748e-01	9.958395e-01	9.998046e-01
	std.	0.000000e+00	1.552235e-02	9.087680e-02	4.742975e-03	1.792805e-01	0.000000e+00	3.885113e-04	3.235526e-04	2.327347e-07	2.583823e-06	1.074986e-02	2.291253e-03	4.031810e-04

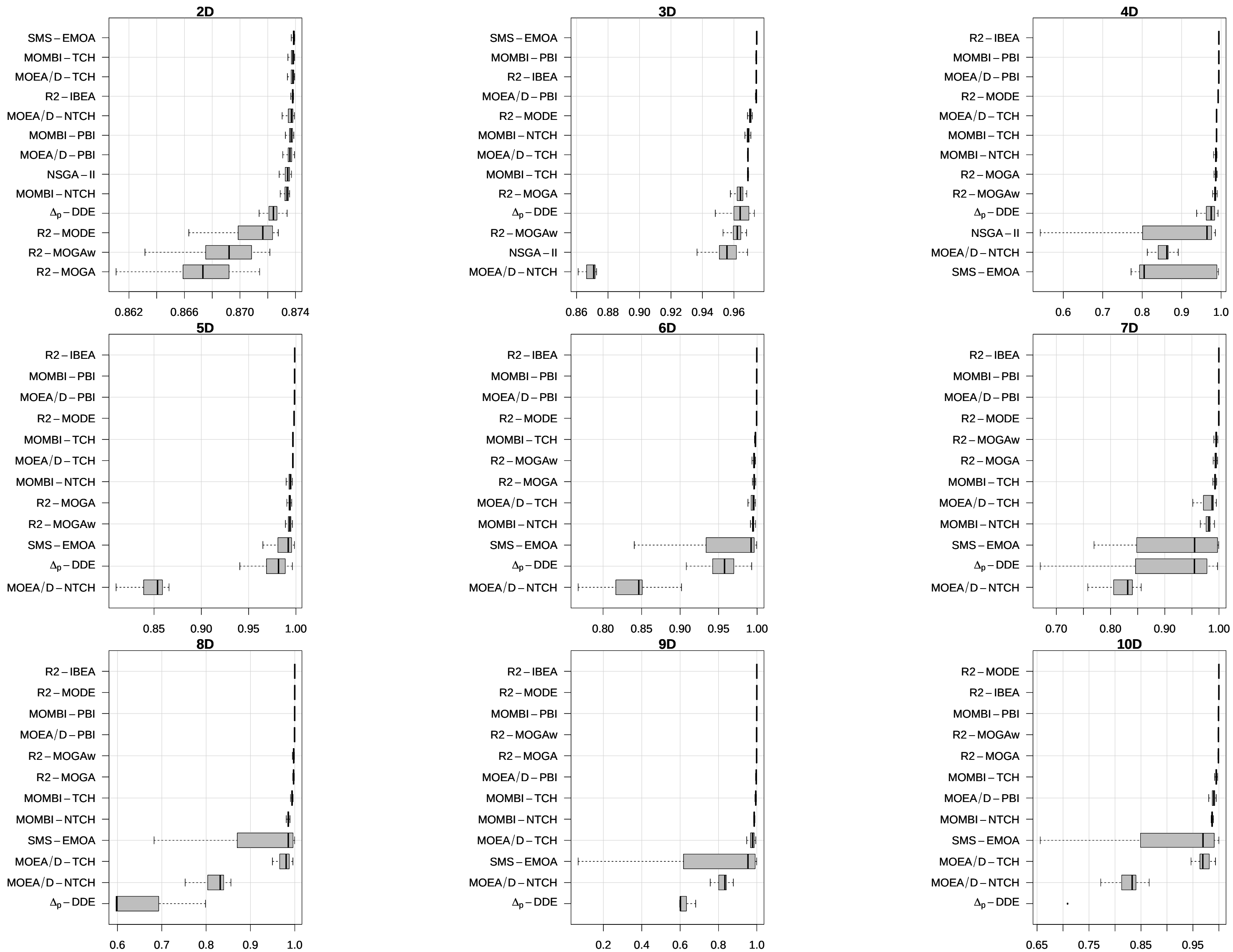


Figure C.1: Box-plot of hypervolume indicator values for different optimizers on the DTLZ1 test problem.

Table C.2: Wilcoxon rank sum test applied to hypervolume indicator values on the DTLZ1 test problem. The table contains for each pair of optimizers O_R (row) and O_C (column) the p-values with respect to the alternative hypothesis that the indicator values for O_R are significantly better than those for O_C.

Optimizer	**NSGA-II**	**MOEA/D TCH**	**MOEA/D NTCH**	**MOEA/D PBI**	**SMS-EMOA**	**Δ_p-DDE**	**R2-MOGA**	**R2-MOGAw**	**R2-MODE**	**R2-IBEA**	**MOMBI TCH**	**MOMBI NTCH**	**MOMBI PBI**
2D													
NSGA-II	–	> 0.05	> 0.05	> 0.05	> 0.05	4.95e − 29	1.28e − 34	1.28e − 34	4.64e − 34	> 0.05	> 0.05	9.31e − 03	> 0.05
MOEA/D-TCH	2.40e − 26	–	4.76e − 07	9.76e − 12	> 0.05	3.27e − 32	1.32e − 34	1.89e − 34	6.63e − 34	> 0.05	> 0.05	6.60e − 30	9.79e − 08
MOEA/D-NTCH	1.70e − 09	> 0.05	–	4.24e − 02	> 0.05	1.86e − 25	2.31e − 32	1.55e − 31	1.12e − 29	> 0.05	> 0.05	1.35e − 12	> 0.05
MOEA/D-PBI	3.54e − 08	> 0.05	> 0.05	–	> 0.05	2.12e − 31	1.28e − 34	1.28e − 34	1.78e − 34	> 0.05	> 0.05	9.49e − 14	> 0.05
SMS-EMOA	3.09e − 32	4.06e − 08	1.77e − 19	1.56e − 22	–	1.52e − 33	1.28e − 34	1.28e − 34	1.28e − 34	1.70e − 13	1.07e − 07	1.87e − 33	4.30e − 21
Δ_p-DDE	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	–	3.05e − 34	7.61e − 31	2.93e − 11	> 0.05	> 0.05	> 0.05	> 0.05
R2-MOGA	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	–	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05
R2-MOGAw	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	3.29e − 07	–	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05
R2-MODE	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	4.85e − 18	4.55e − 10	–	> 0.05	> 0.05	> 0.05	> 0.05
R2-IBEA	1.24e − 32	> 0.05	6.28e − 08	2.05e − 16	> 0.05	2.51e − 33	1.28e − 34	1.28e − 34	1.28e − 34	–	> 0.05	4.50e − 34	9.01e − 11
MOMBI-TCH	3.93e − 26	> 0.05	4.46e − 08	2.28e − 12	> 0.05	3.47e − 33	1.28e − 34	1.28e − 34	1.28e − 34	> 0.05	–	5.91e − 30	1.03e − 08
MOMBI-NTCH	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	4.82e − 29	1.28e − 34	1.32e − 34	4.12e − 34	> 0.05	> 0.05	–	> 0.05
MOMBI-PBI	3.96e − 15	> 0.05	> 0.05	4.34e − 03	> 0.05	1.89e − 32	1.28e − 34	1.28e − 34	1.28e − 34	> 0.05	> 0.05	1.39e − 20	–
3D													
NSGA-II	–	> 0.05	5.05e − 32	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05
MOEA/D-TCH	2.26e − 34	–	1.28e − 34	> 0.05	> 0.05	1.47e − 07	1.36e − 34	1.28e − 34	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05
MOEA/D-NTCH	> 0.05	> 0.05	–	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05
MOEA/D-PBI	1.28e − 34	1.28e − 34	1.28e − 34	–	> 0.05	1.28e − 34	1.28e − 34	1.28e − 34	1.28e − 34	> 0.05	1.28e − 34	1.28e − 34	> 0.05
SMS-EMOA	1.28e − 34	1.28e − 34	1.28e − 34	3.25e − 31	–	1.28e − 34	1.28e − 34	1.28e − 34	1.28e − 34	6.70e − 33	1.28e − 34	1.28e − 34	1.72e − 29
Δ_p-DDE	6.97e − 13	> 0.05	1.61e − 33	> 0.05	> 0.05	–	> 0.05	1.04e − 03	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05
R2-MOGA	2.44e − 17	> 0.05	5.38e − 34	> 0.05	> 0.05	> 0.05	–	8.29e − 06	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05
R2-MOGAw	4.09e − 10	> 0.05	2.44e − 33	> 0.05	> 0.05	> 0.05	> 0.05	–	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05
R2-MODE	1.78e − 34	1.32e − 29	1.28e − 34	> 0.05	> 0.05	4.58e − 16	1.68e − 34	1.40e − 34	–	> 0.05	2.40e − 28	2.62e − 20	> 0.05
R2-IBEA	1.28e − 34	1.28e − 34	1.28e − 34	> 0.05	> 0.05	1.28e − 34	1.28e − 34	1.28e − 34	1.28e − 34	–	1.28e − 34	1.28e − 34	> 0.05
MOMBI-TCH	2.07e − 34	> 0.05	1.28e − 34	> 0.05	> 0.05	1.01e − 07	1.28e − 34	1.28e − 34	> 0.05	> 0.05	–	> 0.05	> 0.05
MOMBI-NTCH	5.71e − 33	> 0.05	1.28e − 34	> 0.05	> 0.05	1.97e − 08	1.22e − 32	1.31e − 33	> 0.05	> 0.05	> 0.05	–	> 0.05
MOMBI-PBI	1.28e − 34	1.28e − 34	1.28e − 34	6.80e − 03	> 0.05	1.28e − 34	1.28e − 34	1.28e − 34	1.28e − 34	8.86e − 03	1.28e − 34	1.28e − 34	–
4D													
NSGA-II	–	> 0.05	3.43e − 04	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05
MOEA/D-TCH	1.28e − 34	–	1.28e − 34	> 0.05	9.23e − 09	2.82e − 27	4.65e − 25	2.68e − 28	> 0.05	> 0.05	5.25e − 04	2.03e − 10	> 0.05
MOEA/D-NTCH	> 0.05	> 0.05	–	> 0.05	3.98e − 02	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05
MOEA/D-PBI	1.28e − 34	1.28e − 34	1.28e − 34	–	1.28e − 34	1.28e − 34	1.28e − 34	1.28e − 34	1.28e − 34	> 0.05	1.28e − 34	1.28e − 34	> 0.05
SMS-EMOA	> 0.05	> 0.05	> 0.05	> 0.05	–	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05
Δ_p-DDE	4.52e − 08	> 0.05	1.84e − 34	> 0.05	3.86e − 06	–	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05
R2-MOGA	2.23e − 33	> 0.05	1.28e − 34	> 0.05	5.04e − 08	1.42e − 17	–	5.13e − 06	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05
R2-MOGAw	2.48e − 30	> 0.05	1.28e − 34	> 0.05	3.55e − 07	1.76e − 13	> 0.05	–	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05
R2-MODE	1.28e − 34	1.28e − 34	1.28e − 34	> 0.05	1.78e − 20	2.55e − 34	1.28e − 34	1.28e − 34	–	> 0.05	1.28e − 34	1.28e − 34	> 0.05
R2-IBEA	1.28e − 34	1.28e − 34	1.28e − 34	2.71e − 20	1.28e − 34	1.28e − 34	1.28e − 34	1.28e − 34	1.28e − 34	–	1.28e − 34	1.28e − 34	1.27e − 09
MOMBI-TCH	1.28e − 34	> 0.05	1.28e − 34	> 0.05	8.60e − 09	2.54e − 27	2.57e − 24	7.24e − 28	> 0.05	> 0.05	–	1.62e − 09	> 0.05
MOMBI-NTCH	9.39e − 33	> 0.05	1.28e − 34	> 0.05	4.11e − 08	2.19e − 19	9.22e − 03	8.36e − 08	> 0.05	> 0.05	> 0.05	–	> 0.05
MOMBI-PBI	1.28e − 34	1.28e − 34	1.28e − 34	4.40e − 06	1.28e − 34	1.28e − 34	1.28e − 34	1.28e − 34	1.28e − 34	> 0.05	1.28e − 34	1.28e − 34	–

Table C.3: Wilcoxon rank sum test applied to hypervolume indicator values on the DTLZ1 test problem (continuation).

Optimizer	NSGA-II	MOEA/D TCH	MOEA/D NTCH	MOEA/D PBI	SMS-EMOA	Δ_p-DDE	R2-MOGA	R2-MOGAw	R2-MODE	R2-IBEA	MOMBI TCH	MOMBI NTCH	MOMBI PBI
							5D						
NSGA-II	–	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05
MOEA/D-TCH	2.82e − 39	–	1.87e − 34	> 0.05	2.30e − 17	2.10e − 33	6.17e − 26	1.52e − 26	> 0.05	> 0.05	> 0.05	8.19e − 26	> 0.05
MOEA/D-NTCH	3.61e − 39	> 0.05	–	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05
MOEA/D-PBI	2.82e − 39	1.28e − 34	1.87e − 34	–	2.07e − 34	1.28e − 34	1.28e − 34	1.28e − 34	7.34e − 32	> 0.05	1.28e − 34	1.28e − 34	> 0.05
SMS-EMOA	2.82e − 39	> 0.05	1.15e − 22	> 0.05	–	7.46e − 07	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05
Δ_p-DDE	2.82e − 39	> 0.05	1.87e − 34	> 0.05	> 0.05	–	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05
R2-MOGA	2.82e − 39	> 0.05	1.87e − 34	> 0.05	1.67e − 03	1.25e − 28	–	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05
R2-MOGAw	2.82e − 39	> 0.05	1.87e − 34	> 0.05	2.89e − 03	5.60e − 27	> 0.05	–	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05
R2-MODE	2.82e − 39	1.28e − 34	1.87e − 34	> 0.05	1.69e − 31	1.28e − 34	1.28e − 34	1.28e − 34	–	> 0.05	1.28e − 34	1.28e − 34	> 0.05
R2-IBEA	2.82e − 39	1.28e − 34	1.87e − 34	1.28e − 34	1.28e − 34	1.28e − 34	1.28e − 34	1.28e − 34	1.28e − 34	–	1.28e − 34	1.28e − 34	5.14e − 34
MOMBI-TCH	2.82e − 39	6.23e − 20	1.87e − 34	> 0.05	8.38e − 21	1.28e − 34	1.28e − 34	1.28e − 34	> 0.05	> 0.05	–	1.28e − 34	> 0.05
MOMBI-NTCH	2.82e − 39	> 0.05	1.87e − 34	> 0.05	3.40e − 04	3.70e − 28	3.28e − 02	> 0.05	> 0.05	> 0.05	> 0.05	–	> 0.05
MOMBI-PBI	2.82e − 39	1.28e − 34	1.87e − 34	1.18e − 26	1.32e − 34	1.28e − 34	1.28e − 34	1.28e − 34	1.28e − 34	> 0.05	1.28e − 34	1.28e − 34	–
							6D						
NSGA-II	–	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05
MOEA/D-TCH	2.82e − 39	–	1.87e − 34	> 0.05	6.27e − 04	1.54e − 32	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05
MOEA/D-NTCH	3.61e − 39	> 0.05	–	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05
MOEA/D-PBI	2.82e − 39	1.28e − 34	1.87e − 34	–	1.92e − 33	1.28e − 34	1.28e − 34	1.28e − 34	2.49e − 11	> 0.05	1.28e − 34	1.28e − 34	> 0.05
SMS-EMOA	2.82e − 39	> 0.05	9.09e − 21	> 0.05	–	3.45e − 08	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05
Δ_p-DDE	2.82e − 39	> 0.05	2.77e − 34	> 0.05	> 0.05	–	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05
R2-MOGA	2.82e − 39	5.19e − 07	1.87e − 34	> 0.05	8.76e − 10	1.44e − 34	–	> 0.05	> 0.05	> 0.05	> 0.05	1.04e − 11	> 0.05
R2-MOGAw	2.82e − 39	7.79e − 07	1.87e − 34	> 0.05	4.03e − 10	1.40e − 34	> 0.05	–	> 0.05	> 0.05	> 0.05	3.29e − 11	> 0.05
R2-MODE	2.82e − 39	1.28e − 34	1.87e − 34	> 0.05	2.51e − 33	1.28e − 34	1.28e − 34	1.28e − 34	–	> 0.05	1.28e − 34	1.28e − 34	> 0.05
R2-IBEA	2.82e − 39	1.28e − 34	1.87e − 34	1.28e − 34	1.28e − 34	1.28e − 34	1.28e − 34	1.28e − 34	1.28e − 34	–	1.28e − 34	1.28e − 34	1.28e − 34
MOMBI-TCH	2.82e − 39	7.46e − 29	1.87e − 34	> 0.05	2.83e − 22	1.28e − 34	6.33e − 26	3.61e − 25	> 0.05	> 0.05	–	1.30e − 30	> 0.05
MOMBI-NTCH	2.82e − 39	> 0.05	1.87e − 34	> 0.05	5.30e − 05	2.96e − 34	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	–	> 0.05
MOMBI-PBI	2.82e − 39	1.28e − 34	1.87e − 34	8.06e − 28	1.78e − 34	1.28e − 34	1.28e − 34	1.28e − 34	4.24e − 34	> 0.05	1.28e − 34	1.28e − 34	–
							7D						
NSGA-II	–	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05
MOEA/D-TCH	2.82e − 39	–	1.28e − 34	> 0.05	> 0.05	1.50e − 14	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	1.43e − 04	> 0.05
MOEA/D-NTCH	2.82e − 39	> 0.05	–	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05
MOEA/D-PBI	2.82e − 39	1.28e − 34	1.28e − 34	–	1.28e − 34	1.87e − 34	1.28e − 34	1.28e − 34	1.04e − 20	> 0.05	1.28e − 34	1.28e − 34	> 0.05
SMS-EMOA	2.82e − 39	> 0.05	1.23e − 19	> 0.05	–	8.00e − 03	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05
Δ_p-DDE	3.61e − 39	> 0.05	1.15e − 10	> 0.05	> 0.05	–	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05
R2-MOGA	2.82e − 39	6.00e − 28	1.28e − 34	> 0.05	1.43e − 03	4.53e − 28	–	> 0.05	> 0.05	> 0.05	2.26e − 08	2.71e − 34	> 0.05
R2-MOGAw	2.82e − 39	4.23e − 30	1.28e − 34	> 0.05	2.26e − 04	7.07e − 29	6.82e − 03	–	> 0.05	> 0.05	9.75e − 15	2.13e − 34	> 0.05
R2-MODE	2.82e − 39	1.28e − 34	1.28e − 34	> 0.05	1.40e − 34	1.87e − 34	1.28e − 34	1.28e − 34	–	> 0.05	1.28e − 34	1.28e − 34	> 0.05
R2-IBEA	2.82e − 39	1.28e − 34	1.28e − 34	8.39e − 34	1.28e − 34	1.87e − 34	1.28e − 34	1.28e − 34	1.28e − 34	–	1.28e − 34	1.28e − 34	1.48e − 30
MOMBI-TCH	2.82e − 39	1.01e − 21	1.28e − 34	> 0.05	8.24e − 03	2.90e − 27	> 0.05	> 0.05	> 0.05	> 0.05	–	1.00e − 33	> 0.05
MOMBI-NTCH	2.82e − 39	> 0.05	1.28e − 34	> 0.05	> 0.05	3.45e − 12	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	–	> 0.05
MOMBI-PBI	2.82e − 39	1.28e − 34	1.28e − 34	4.27e − 18	1.28e − 34	1.87e − 34	1.28e − 34	1.28e − 34	1.53e − 34	> 0.05	1.28e − 34	1.28e − 34	–

Table C.4: Wilcoxon rank sum test applied to hypervolume indicator values on the DTLZ1 test problem (continuation).

Optimizer	**NSGA-II**	**MOEA/D TCH**	**MOEA/D NTCH**	**MOEA/D PBI**	**SMS-EMOA**	**Δ_p-DDE**	**R2-MOGA**	**R2-MOGAw**	**R2-MODE**	**R2-IBEA**	**MOMBI TCH**	**MOMBI NTCH**	**MOMBI PBI**
8D													
NSGA-II	–	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05
MOEA/D-TCH	2.82e-39	–	1.28e-34	> 0.05	> 0.05	4.69e-24	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05
MOEA/D-NTCH	2.82e-39	> 0.05	–	> 0.05	> 0.05	5.48e-12	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05
MOEA/D-PBI	2.82e-39	1.28e-34	1.28e-34	–	3.57e-30	1.87e-34	1.28e-34	1.84e-34	> 0.05	> 0.05	1.28e-34	1.28e-34	> 0.05
SMS-EMOA	5.97e-39	> 0.05	1.70e-13	> 0.05	–	4.61e-16	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05
Δ_p-DDE	3.61e-39	> 0.05	> 0.05	> 0.05	> 0.05	–	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05
R2-MOGA	2.82e-39	1.53e-34	1.28e-34	> 0.05	7.98e-18	4.07e-33	–	> 0.05	> 0.05	> 0.05	7.65e-33	1.28e-34	> 0.05
R2-MOGAw	2.82e-39	1.78e-34	1.28e-34	> 0.05	2.26e-18	3.31e-33	> 0.05	–	> 0.05	> 0.05	1.30e-31	1.36e-34	> 0.05
R2-MODE	2.81e-39	1.28e-34	1.28e-34	1.28e-34	4.04e-34	1.87e-34	1.28e-34	1.28e-34	–	> 0.05	1.28e-34	1.28e-34	4.85e-33
R2-IBEA	2.82e-39	1.28e-34	1.28e-34	1.28e-34	4.04e-34	1.87e-34	1.28e-34	1.28e-34	4.42e-34	–	1.28e-34	1.28e-34	1.28e-34
MOMBI-TCH	2.82e-39	8.48e-30	1.28e-34	> 0.05	3.13e-05	1.01e-29	> 0.05	> 0.05	> 0.05	> 0.05	–	2.12e-32	> 0.05
MOMBI-NTCH	2.82e-39	2.67e-04	1.28e-34	> 0.05	> 0.05	4.62e-26	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	–	> 0.05
MOMBI-PBI	2.82e-39	1.28e-34	1.28e-34	7.45e-32	2.77e-33	1.87e-34	1.28e-34	1.28e-34	> 0.05	> 0.05	1.28e-34	1.28e-34	–
9D													
NSGA-II	–	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05
MOEA/D-TCH	2.82e-39	–	1.28e-34	> 0.05	1.78e-02	1.93e-32	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05
MOEA/D-NTCH	2.82e-39	> 0.05	–	> 0.05	> 0.05	1.69e-22	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05
MOEA/D-PBI	2.82e-39	2.01e-34	1.28e-34	–	1.01e-21	1.93e-34	> 0.05	> 0.05	> 0.05	> 0.05	2.13e-22	1.95e-34	> 0.05
SMS-EMOA	4.64e-39	> 0.05	1.32e-03	> 0.05	–	4.31e-08	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05
Δ_p-DDE	3.61e-39	> 0.05	> 0.05	> 0.05	> 0.05	–	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05
R2-MOGA	2.82e-39	1.28e-34	1.28e-34	2.82e-27	5.82e-29	1.87e-34	–	> 0.05	> 0.05	> 0.05	1.44e-34	1.28e-34	> 0.05
R2-MOGAw	2.82e-39	1.28e-34	1.28e-34	1.66e-27	1.06e-29	1.87e-34	7.37e-03	–	> 0.05	> 0.05	1.44e-34	1.28e-34	> 0.05
R2-MODE	2.79e-39	1.27e-34	1.27e-34	1.27e-34	2.72e-34	1.86e-34	1.27e-34	1.27e-34	–	> 0.05	1.27e-34	1.27e-34	1.27e-34
R2-IBEA	2.80e-39	1.27e-34	1.27e-34	1.27e-34	2.73e-34	1.86e-34	1.27e-34	1.27e-34	1.66e-09	–	1.27e-34	1.27e-34	1.27e-34
MOMBI-TCH	2.82e-39	9.39e-33	1.28e-34	> 0.05	2.02e-14	3.13e-34	> 0.05	> 0.05	> 0.05	> 0.05	–	8.85e-33	> 0.05
MOMBI-NTCH	2.82e-39	3.31e-14	1.28e-34	> 0.05	6.57e-05	3.51e-33	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	–	> 0.05
MOMBI-PBI	2.82e-39	1.28e-34	1.28e-34	1.28e-34	3.40e-34	1.87e-34	1.95e-34	1.73e-34	> 0.05	> 0.05	1.28e-34	1.28e-34	–
10D													
NSGA-II	–	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05
MOEA/D-TCH	7.88e-20	–	1.36e-34	> 0.05	> 0.05	4.48e-02	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05
MOEA/D-NTCH	7.88e-20	> 0.05	–	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05
MOEA/D-PBI	7.88e-20	2.84e-24	1.28e-34	–	6.78e-07	4.48e-02	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	1.76e-07	> 0.05
SMS-EMOA	5.59e-19	> 0.05	6.89e-12	> 0.05	–	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05
Δ_p-DDE	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	–	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05
R2-MOGA	7.88e-20	1.28e-34	1.28e-34	1.28e-34	9.45e-29	4.48e-02	–	> 0.05	> 0.05	> 0.05	1.32e-34	1.28e-34	> 0.05
R2-MOGAw	7.88e-20	1.28e-34	1.28e-34	1.28e-34	7.70e-29	4.48e-02	> 0.05	–	> 0.05	> 0.05	1.28e-34	1.28e-34	> 0.05
R2-MODE	7.29e-20	1.11e-34	1.11e-34	1.11e-34	3.84e-33	4.36e-02	1.11e-34	1.11e-34	–	1.63e-34	1.11e-34	1.11e-34	1.11e-34
R2-IBEA	7.86e-20	1.27e-34	1.27e-34	1.27e-34	4.47e-33	4.47e-02	1.27e-34	1.27e-34	> 0.05	–	1.27e-34	1.27e-34	1.27e-34
MOMBI-TCH	7.88e-20	9.39e-33	1.36e-34	1.30e-27	1.62e-17	4.48e-02	> 0.05	> 0.05	> 0.05	> 0.05	–	6.79e-30	> 0.05
MOMBI-NTCH	7.88e-20	8.40e-22	1.28e-34	> 0.05	3.85e-04	4.48e-02	> 0.05	> 0.05	> 0.05	> 0.05	> 0.05	–	> 0.05
MOMBI-PBI	7.88e-20	1.28e-34	1.28e-34	1.28e-34	6.85e-29	4.48e-02	1.65e-03	4.37e-02	> 0.05	> 0.05	1.32e-34	1.28e-34	–

Table C.5: Comparison of R2 indicator values for different optimizers on the DTLZ1 test problem.

Dim.	Stat.	NSGA-II	MOEA/D TCH	MOEA/D NTCH	MOEA/D PBI	SMS-EMOA	Δ_p-DDE	R2-MOGA	R2-MOGAw	R2-MODE	R2-IBEA	MOMBI TCH	MOMBI NTCH	MOMBI PBI
2D	med.	8.385074e − 02	8.372010e − 02	8.375051e − 02	8.376840e − 02	8.370542e − 02	8.368671e − 02	8.440180e − 02	8.410993e − 02	8.416171e − 02	8.373354e − 02	8.371485e − 02	8.383756e − 02	8.373073e − 02
	avg.	8.387052e − 02	8.373446e − 02	8.391282e − 02	8.378113e − 02	8.371807e − 02	8.369134e − 02	8.443781e − 02	8.413814e − 02	8.418518e − 02	8.374013e − 02	8.373025e − 02	8.385909e − 02	8.375069e − 02
	min.	8.375748e − 02	8.368085e − 02	8.368644e − 02	8.368010e − 02	8.368124e − 02	8.367663e − 02	8.418900e − 02	8.395160e − 02	8.402386e − 02	8.370817e − 02	8.367925e − 02	8.375705e − 02	8.367832e − 02
	max.	8.425286e − 02	8.384963e − 02	9.174951e − 02	8.401523e − 02	8.388412e − 02	8.379699e − 02	8.480666e − 02	8.477390e − 02	8.455391e − 02	8.386054e − 02	8.395013e − 02	8.425462e − 02	8.398607e − 02
	std.	8.144398e − 05	4.367285e − 05	8.413448e − 04	7.285276e − 05	3.498272e − 05	1.995449e − 05	1.361729e − 04	1.276486e − 04	9.546413e − 05	2.402972e − 05	5.199513e − 05	6.952766e − 05	6.937738e − 05
3D	med.	3.511511e − 02	3.302707e − 02	5.608644e − 02	3.231443e − 02	3.222944e − 02	3.295353e − 02	3.334104e − 02	3.287088e − 02	3.313518e − 02	3.242773e − 02	3.300843e − 02	3.339101e − 02	3.228973e − 02
	avg.	3.571466e − 02	3.304185e − 02	5.625453e − 02	3.233669e − 02	3.223257e − 02	3.301123e − 02	3.332621e − 02	3.289627e − 02	3.314273e − 02	3.243327e − 02	3.302590e − 02	3.344065e − 02	3.230917e − 02
	min.	3.336875e − 02	3.295928e − 02	3.480163e − 02	3.223014e − 02	3.210057e − 02	3.248863e − 02	3.292960e − 02	3.256125e − 02	3.280510e − 02	3.227766e − 02	3.292144e − 02	3.299791e − 02	3.222137e − 02
	max.	4.206462e − 02	3.337677e − 02	7.329071e − 02	3.258436e − 02	3.240561e − 02	3.366187e − 02	3.378800e − 02	3.376255e − 02	3.351496e − 02	3.259226e − 02	3.322313e − 02	3.426621e − 02	3.257814e − 02
	std.	1.886720e − 03	6.001562e − 05	7.064818e − 03	8.352625e − 05	5.590796e − 05	2.004498e − 04	1.901797e − 04	1.998662e − 04	1.212611e − 04	5.020220e − 05	5.538035e − 05	2.292650e − 04	7.026704e − 05
4D	med.	2.380465e − 02	2.067428e − 02	4.327824e − 02	1.815940e − 02	3.612566e − 02	1.933165e − 02	1.849178e − 02	1.812627e − 02	1.841000e − 02	1.790737e − 02	2.068479e − 02	2.098440e − 02	1.811428e − 02
	avg.	3.672688e − 02	2.067669e − 02	4.558573e − 02	1.818322e − 02	3.086051e − 02	1.943022e − 02	1.850923e − 02	1.815813e − 02	1.841859e − 02	1.790918e − 02	2.069069e − 02	2.119110e − 02	1.813441e − 02
	min.	1.996967e − 02	2.056859e − 02	2.943675e − 02	1.807106e − 02	1.826573e − 02	1.841516e − 02	1.820051e − 02	1.783888e − 02	1.813560e − 02	1.777702e − 02	2.048579e − 02	2.034299e − 02	1.806582e − 02
	max.	1.011482e − 01	2.080705e − 02	9.839260e − 02	1.854938e − 02	3.817969e − 02	2.194369e − 02	1.897667e − 02	1.861518e − 02	1.875237e − 02	1.803716e − 02	2.085206e − 02	2.320697e − 02	1.853196e − 02
	std.	2.137609e − 02	4.584309e − 05	7.891262e − 03	8.947101e − 05	7.854645e − 03	5.896343e − 04	1.439326e − 04	1.778563e − 04	1.121107e − 04	5.785786e − 05	6.243442e − 05	6.375727e − 04	7.275515e − 05
5D	med.	1.538539e + 00	1.441970e − 02	3.586743e − 02	1.241561e − 02	1.608386e − 02	1.316910e − 02	1.236398e − 02	1.207117e − 02	1.233718e − 02	1.189470e − 02	1.433294e − 02	1.552536e − 02	1.220159e − 02
	avg.	1.645142e + 00	1.459830e − 02	3.963484e − 02	1.241456e − 02	1.773069e − 02	1.329807e − 02	1.236966e − 02	1.209490e − 02	1.234055e − 02	1.189087e − 02	1.434022e − 02	1.561340e − 02	1.221597e − 02
	min.	6.267370e − 01	1.415217e − 02	2.632329e − 02	1.212947e − 02	1.247020e − 02	1.235684e − 02	1.198295e − 02	1.178485e − 02	1.209160e − 02	1.177697e − 02	1.419539e − 02	1.453311e − 02	1.214845e − 02
	max.	2.987237e + 00	1.700467e − 02	1.905702e − 01	1.274572e − 02	4.496842e − 02	1.509710e − 02	1.268345e − 02	1.250317e − 02	1.270481e − 02	1.201656e − 02	1.448524e − 02	1.739552e − 02	1.247271e − 02
	std.	5.037213e − 01	5.825554e − 04	1.681758e − 02	1.351271e − 04	4.980183e − 03	5.785307e − 04	1.311621e − 04	1.710067e − 04	1.090622e − 04	4.673402e − 05	4.537194e − 05	5.895229e − 04	5.280573e − 05
6D	med.	3.368002e + 00	1.263114e − 02	3.055603e − 02	8.819247e − 03	1.251661e − 02	1.099805e − 02	8.720381e − 03	8.516255e − 03	8.676293e − 03	8.338173e − 03	1.145309e − 02	1.252880e − 02	8.624612e − 03
	avg.	3.407155e + 00	1.282812e − 02	3.729839e − 02	8.830817e − 03	1.433844e − 02	1.099722e − 02	8.734253e − 03	8.530570e − 03	8.682295e − 03	8.338251e − 03	1.163436e − 02	1.260640e − 02	8.639366e − 03
	min.	1.863253e + 00	1.107264e − 02	2.200590e − 02	8.585989e − 03	8.617235e − 03	9.716002e − 03	8.457257e − 03	8.256013e − 03	8.493021e − 03	8.258753e − 03	1.081280e − 02	1.111303e − 02	8.585540e − 03
	max.	5.389775e + 00	1.778660e − 02	4.679480e − 01	9.072356e − 03	3.302069e − 02	1.310498e − 02	8.997348e − 03	8.970798e − 03	8.905098e − 03	8.413354e − 03	1.268313e − 02	1.570174e − 02	8.823907e − 03
	std.	7.306827e − 01	1.109950e − 03	4.416550e − 02	1.157802e − 04	4.568801e − 03	7.675765e − 04	1.096513e − 04	1.462861e − 04	7.627201e − 05	3.446036e − 05	4.502356e − 04	6.082169e − 04	4.373760e − 05
7D	med.	4.074695e + 00	1.262226e − 02	2.783701e − 02	6.955768e − 03	1.391703e − 02	9.081126e − 03	7.394306e − 03	7.150244e − 03	7.075159e − 03	6.704738e − 03	1.199106e − 02	1.356857e − 02	6.843416e − 03
	avg.	3.982829e + 00	1.326928e − 02	2.883675e − 02	6.970629e − 03	1.281230e − 02	1.513928e − 02	7.404865e − 03	7.167509e − 03	7.099456e − 03	6.707208e − 03	1.212769e − 02	1.371012e − 02	6.853955e − 03
	min.	1.238641e + 00	1.094994e − 02	1.798826e − 02	6.814987e − 03	7.538943e − 03	7.390429e − 03	7.069642e − 03	6.556944e − 03	6.884335e − 03	6.630220e − 03	1.084126e − 02	1.263654e − 02	6.812435e − 03
	max.	5.676402e + 00	1.728792e − 02	3.489126e − 02	7.241262e − 03	2.654798e − 02	7.160176e − 02	7.898214e − 03	7.870792e − 03	7.556828e − 03	6.833093e − 03	1.381652e − 02	1.552167e − 02	7.000768e − 03
	std.	8.833205e − 01	1.480894e − 03	3.135724e − 03	7.816529e − 05	4.004891e − 03	1.282587e − 02	1.540721e − 04	2.597167e − 04	1.305124e − 04	4.343099e − 05	8.185360e − 04	6.455343e − 04	3.606924e − 05
8D	med.	4.593328e + 00	1.166084e − 02	2.450944e − 02	5.409252e − 03	1.008923e − 02	2.080289e − 02	5.453639e − 03	5.386111e − 03	5.329522e − 03	5.053873e − 03	1.034324e − 02	1.129017e − 02	5.225100e − 03
	avg.	4.434247e + 00	1.206464e − 02	2.634852e − 02	5.425904e − 03	1.213666e − 02	2.380381e − 02	5.464507e − 03	5.404871e − 03	5.335763e − 03	5.056324e − 03	1.031187e − 02	1.137417e − 02	5.231620e − 03
	min.	2.164358e + 00	9.188365e − 03	1.791018e − 02	5.276984e − 03	5.885885e − 03	6.435851e − 03	5.297021e − 03	5.042237e − 03	5.172752e − 03	4.974724e − 03	9.143791e − 03	1.053461e − 02	5.178599e − 03
	max.	6.042133e + 00	1.514940e − 02	8.254573e − 02	5.790345e − 03	3.590651e − 02	5.884447e − 01	5.699419e − 03	5.898154e − 03	5.525670e − 03	5.131232e − 03	1.146526e − 02	1.233845e − 02	5.343359e − 03
	std.	8.506108e − 01	1.471225e − 03	7.030882e − 03	9.557108e − 05	5.770139e − 03	5.726075e − 02	7.481486e − 05	1.853366e − 04	6.107140e − 05	2.685240e − 05	6.187582e − 04	3.190228e − 04	3.138622e − 05
9D	med.	4.122397e + 00	1.056012e − 02	2.135718e − 02	4.559128e − 03	1.062616e − 02	1.716218e − 02	4.210159e − 03	4.220869e − 03	4.149619e − 03	3.975563e − 03	8.959473e − 03	9.959290e − 03	4.036481e − 03
	avg.	3.967654e + 00	1.082990e − 02	2.258146e − 02	4.567771e − 03	1.300223e − 02	1.698132e − 02	4.215019e − 03	4.236080e − 03	4.153083e − 03	3.974557e − 03	8.980014e − 03	9.991739e − 03	4.046453e − 03
	min.	2.027321e + 00	8.615530e − 03	1.442916e − 02	4.255535e − 03	5.789860e − 03	5.944334e − 03	4.101722e − 03	3.966643e − 03	4.079717e − 03	3.940821e − 03	7.789284e − 03	9.581809e − 03	3.976041e − 03
	max.	5.003975e + 00	1.375822e − 02	7.004212e − 02	4.845897e − 03	3.168857e − 02	8.959756e − 02	4.358280e − 03	4.719741e − 03	4.286632e − 03	4.016907e − 03	1.050988e − 02	1.068242e − 02	4.173123e − 03
	std.	6.174998e − 01	1.122155e − 03	5.422224e − 03	1.213968e − 04	6.082589e − 03	8.553691e − 03	4.910179e − 05	1.471167e − 04	3.951595e − 05	1.765324e − 05	5.890921e − 04	2.136280e − 04	4.399161e − 05
10D	med.	3.297547e + 00	1.017124e − 02	1.935478e − 02	4.180140e − 03	1.083809e − 02	1.360416e − 02	3.359203e − 03	3.358274e − 03	3.320644e − 03	3.203359e − 03	7.844122e − 03	8.813942e − 03	3.290140e − 03
	avg.	3.268733e + 00	1.026991e − 02	2.047036e − 02	4.189758e − 03	1.348278e − 02	1.310704e − 02	3.361429e − 03	3.410268e − 03	3.321271e − 03	3.204765e − 03	7.934386e − 03	8.776893e − 03	3.302600e − 03
	min.	2.214025e + 00	7.724546e − 03	1.396950e − 02	3.793774e − 03	5.436434e − 03	7.549345e − 03	3.274146e − 03	3.183213e − 03	3.243797e − 03	3.179371e − 03	6.814512e − 03	7.330872e − 03	3.185148e − 03
	max.	4.079931e + 00	1.457792e − 02	4.865170e − 02	4.862619e − 03	3.862482e − 02	1.438692e − 02	3.612650e − 03	3.776102e − 03	3.387521e − 03	3.243294e − 03	1.630381e − 02	9.313915e − 03	3.484617e − 03
	std.	3.989123e − 01	1.210794e − 03	3.930391e − 03	1.835765e − 04	6.760764e − 03	1.709068e − 03	5.194446e − 05	1.289282e − 04	2.575418e − 05	1.366408e − 05	9.852051e − 04	2.943461e − 04	6.354346e − 05

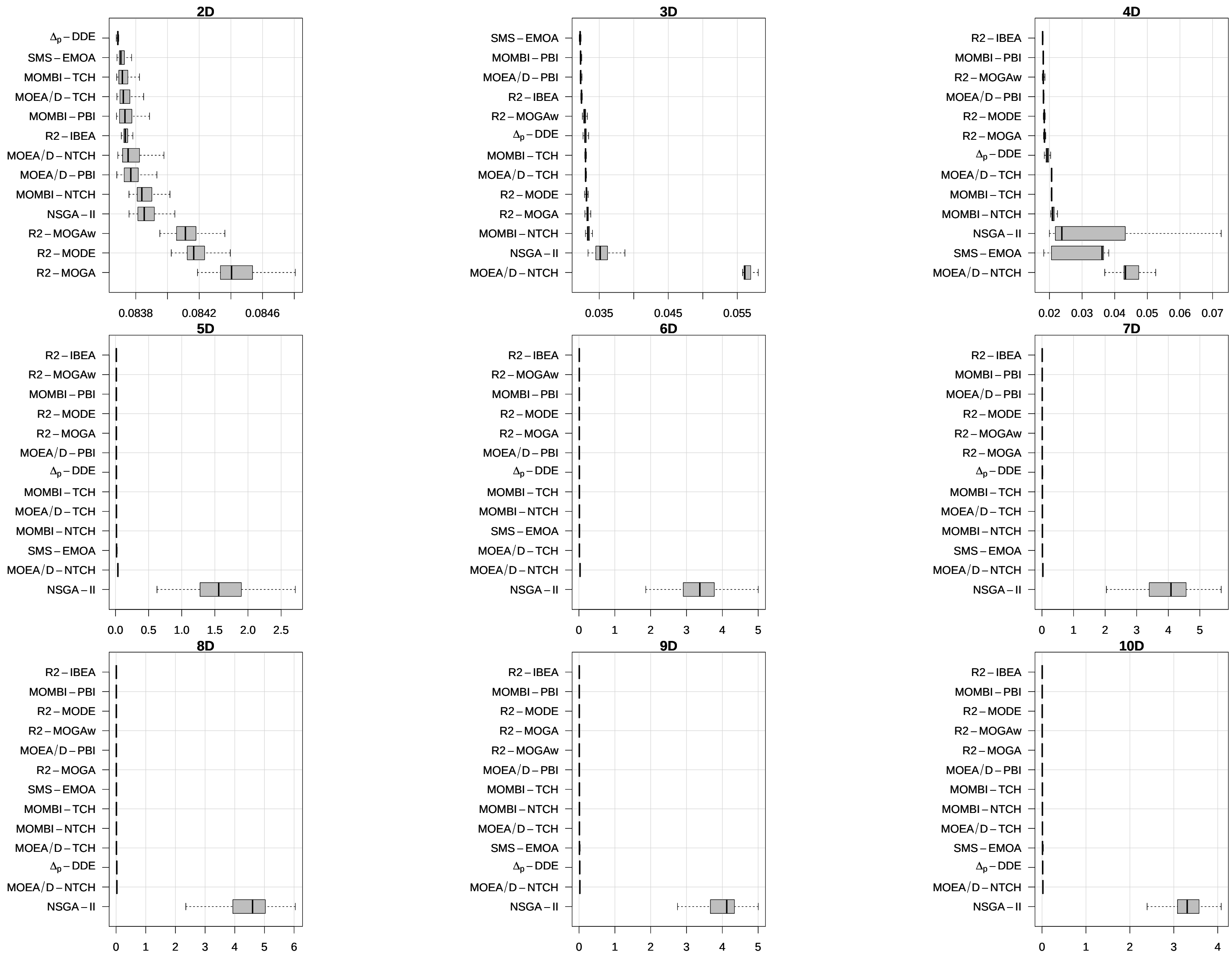


Figure C.2: Box-plot of R2 indicator values for different optimizers on the DTLZ1 test problem.

Table C.6: Comparison of runtime (in milliseconds) for different optimizers on the DTLZ1 test problem.

Dim.	Stat.	NSGA-II	MOEA/D TCH	MOEA/D NTCH	MOEA/D PBI	SMS-EMOA	Δ_p-DDE	R2-MOGA	R2-MOGAw	R2-MODE	R2-IBEA	MOMBI TCH	MOMBI NTCH	MOMBI PBI
2D	med.	1.391130e+02	3.585500e+02	8.890920e+02	8.430090e+02	2.195065e+04	1.008147e+03	2.075753e+03	1.659484e+03	2.008625e+03	5.854420e+05	1.941276e+03	1.495085e+03	2.641578e+03
	avg.	1.397804e+02	3.630220e+02	8.837353e+02	8.298641e+02	2.122329e+04	1.017312e+03	1.998703e+03	1.635794e+03	1.986146e+03	5.895935e+05	1.881835e+03	1.505769e+03	2.622540e+03
	min.	1.306990e+02	3.306930e+02	6.170960e+02	5.426120e+02	1.522593e+04	9.197430e+02	1.444663e+03	1.197029e+03	1.347716e+03	5.513610e+05	1.432049e+03	1.474689e+03	2.186127e+03
	max.	1.525670e+02	4.208590e+02	1.210603e+03	1.064243e+03	2.438248e+04	1.159957e+03	2.760808e+03	2.096304e+03	2.614321e+03	6.352280e+05	2.309959e+03	1.628995e+03	2.978548e+03
	std.	4.793493e+00	1.597084e+01	1.573511e+02	8.379470e+01	2.289796e+03	4.562875e+01	3.722775e+02	2.445412e+02	3.295403e+02	2.304065e+04	2.183163e+02	3.386895e+01	1.461823e+02
3D	med.	2.227720e+02	5.261070e+02	1.236478e+03	9.189210e+02	3.657331e+05	1.173243e+03	2.080344e+03	1.590688e+03	2.101381e+03	5.792150e+05	1.917696e+03	1.839201e+03	2.861312e+03
	avg.	2.309763e+02	5.301123e+02	1.219268e+03	9.111462e+02	3.586857e+05	1.161146e+03	2.065732e+03	1.607572e+03	2.089505e+03	5.828010e+05	1.894798e+03	1.867943e+03	2.856074e+03
	min.	1.901210e+02	3.845070e+02	9.457770e+02	6.842580e+02	2.526093e+05	7.658650e+02	1.448911e+03	1.119894e+03	1.377074e+03	5.283550e+05	1.512564e+03	1.611729e+03	2.353641e+03
	max.	3.477900e+02	8.270200e+02	1.466421e+03	1.392306e+03	4.344593e+05	1.715951e+03	2.557514e+03	2.307402e+03	2.503613e+03	6.270280e+05	2.280909e+03	2.184837e+03	3.134970e+03
	std.	3.382368e+01	9.523016e+01	1.155408e+02	9.675797e+01	3.373515e+04	1.933304e+02	2.642836e+02	1.999351e+02	2.211111e+02	2.150778e+04	1.662622e+02	1.347468e+02	1.322137e+02
4D	med.	3.030930e+02	5.257360e+02	1.342439e+03	9.522530e+02	6.526423e+05	1.211161e+03	2.343130e+03	1.783164e+03	2.487400e+03	4.838380e+05	1.926371e+03	2.118641e+03	3.177616e+03
	avg.	3.116593e+02	5.444823e+02	1.310323e+03	9.510556e+02	7.365434e+05	1.197820e+03	2.291805e+03	1.784849e+03	2.427117e+03	4.846312e+05	1.920583e+03	2.097295e+03	3.160794e+03
	min.	2.497280e+02	3.936580e+02	8.587430e+02	6.923800e+02	2.194115e+05	8.297930e+02	1.673061e+03	1.264352e+03	1.698642e+03	4.822670e+05	1.553307e+03	1.726381e+03	2.686032e+03
	max.	4.326080e+02	1.003001e+03	1.722781e+03	1.294281e+03	1.885482e+06	1.483117e+03	2.724454e+03	2.274551e+03	2.836077e+03	4.951190e+05	2.415609e+03	2.622699e+03	3.420097e+03
	std.	4.486370e+01	1.023649e+02	1.392530e+02	7.939919e+01	4.228194e+05	1.483485e+02	2.410393e+02	2.086896e+02	2.904422e+02	2.807689e+03	1.643593e+02	1.594938e+02	1.290294e+02
5D	med.	4.634650e+02	5.936830e+02	1.716056e+03	9.008690e+02	3.688071e+05	1.966702e+03	2.752027e+03	2.170170e+03	2.952324e+03	6.723700e+05	2.056078e+03	2.316038e+03	3.466887e+03
	avg.	4.656389e+02	5.994942e+02	1.646845e+03	9.293176e+02	4.758052e+05	1.952076e+03	2.728022e+03	2.182887e+03	2.910408e+03	6.719233e+05	2.091996e+03	2.317876e+03	3.470743e+03
	min.	3.604860e+02	4.508890e+02	6.825210e+02	6.944790e+02	2.277904e+05	1.357003e+03	1.965132e+03	1.696206e+03	2.031878e+03	6.638520e+05	1.674114e+03	2.037259e+03	2.865395e+03
	max.	6.387590e+02	9.130550e+02	2.092520e+03	1.910748e+03	1.774725e+06	2.566455e+03	3.170497e+03	2.770856e+03	3.378682e+03	6.787380e+05	2.885268e+03	2.657465e+03	3.845690e+03
	std.	5.204335e+01	7.438336e+01	2.534973e+02	1.595797e+02	3.174793e+05	2.471749e+02	2.888072e+02	2.265143e+02	3.242308e+02	3.054223e+03	2.487035e+02	1.450260e+02	1.994397e+02
6D	med.	6.200720e+02	6.141290e+02	1.848598e+03	9.525350e+02	4.499790e+05	2.803394e+03	2.951671e+03	2.389304e+03	3.135965e+03	7.114200e+05	2.161011e+03	2.292713e+03	3.870049e+03
	avg.	6.165837e+02	6.255701e+02	1.772065e+03	9.484281e+02	4.952557e+05	2.890935e+03	2.900353e+03	2.376442e+03	3.139869e+03	7.108565e+05	2.221818e+03	2.318226e+03	3.865938e+03
	min.	4.665750e+02	4.753340e+02	8.113770e+02	6.754810e+02	2.749547e+05	1.865957e+03	2.019626e+03	1.774967e+03	2.291378e+03	6.820630e+05	1.722768e+03	2.069538e+03	3.343604e+03
	max.	7.517850e+02	8.888790e+02	2.319126e+03	1.544135e+03	2.412645e+06	6.193999e+03	3.673554e+03	3.023985e+03	3.796013e+03	7.193500e+05	3.848331e+03	2.712226e+03	4.422082e+03
	std.	6.442564e+01	7.630287e+01	2.816327e+02	1.517815e+02	2.520657e+05	6.518767e+02	4.336122e+02	2.763189e+02	3.150468e+02	4.755295e+03	3.290843e+02	1.409623e+02	1.848074e+02
7D	med.	5.668250e+02	6.436060e+02	1.862152e+03	1.050558e+03	6.375392e+05	6.320692e+03	2.301871e+03	1.750564e+03	2.328598e+03	3.320390e+05	1.309713e+03	1.622033e+03	2.691921e+03
	avg.	5.592820e+02	6.507151e+02	1.826127e+03	1.018635e+03	9.082189e+05	7.318769e+03	2.274965e+03	1.755827e+03	2.309991e+03	3.316906e+05	1.320385e+03	1.648061e+03	2.688187e+03
	min.	4.345300e+02	4.482360e+02	8.116300e+02	7.495980e+02	3.803580e+05	3.203776e+03	1.576329e+03	1.306602e+03	1.623444e+03	3.237260e+05	1.100182e+03	1.433981e+03	2.240756e+03
	max.	6.920420e+02	9.999210e+02	2.387335e+03	1.203862e+03	3.100206e+06	8.760375e+04	2.709367e+03	2.129320e+03	2.771965e+03	3.409000e+05	1.654394e+03	1.908542e+03	2.982349e+03
	std.	6.004572e+01	8.400877e+01	2.872730e+02	1.222994e+02	5.416824e+05	8.317682e+03	2.320062e+02	1.598826e+02	2.393009e+02	3.332070e+03	1.270077e+02	1.049933e+02	1.351849e+02
8D	med.	8.159570e+02	6.783160e+02	2.077417e+03	1.125317e+03	6.283459e+05	1.500830e+04	3.738893e+03	2.894609e+03	3.572563e+03	1.286300e+06	2.149195e+03	2.437462e+03	4.128187e+03
	avg.	8.189387e+02	6.796730e+02	1.957683e+03	1.089778e+03	7.241444e+05	1.527586e+04	3.610490e+03	2.904420e+03	3.598642e+03	1.262558e+06	2.134565e+03	2.477580e+03	4.158959e+03
	min.	6.388400e+02	5.368020e+02	9.169490e+02	7.878600e+02	4.086652e+05	6.627723e+03	2.466567e+03	2.013319e+03	2.560665e+03	1.180581e+06	1.677838e+03	2.129564e+03	3.952921e+03
	max.	9.931090e+02	1.041771e+03	2.346116e+03	1.308768e+03	2.727483e+06	3.323225e+04	4.229315e+03	3.580988e+03	4.169719e+03	1.320895e+06	2.493505e+03	2.900989e+03	4.630268e+03
	std.	7.387042e+01	6.217066e+01	3.351714e+02	1.107520e+02	3.931621e+05	3.392000e+03	4.045766e+02	3.132332e+02	3.851677e+02	4.956893e+04	1.682012e+02	1.741348e+02	1.264868e+02
9D	med.	1.240255e+03	7.164020e+02	2.389168e+03	1.003383e+03	7.079355e+05	2.045421e+04	5.734188e+03	4.904337e+03	5.663101e+03	2.261923e+06	3.078124e+03	3.663268e+03	5.901198e+03
	avg.	1.231863e+03	7.140057e+02	2.276141e+03	1.019763e+03	6.680845e+05	2.155547e+04	5.572087e+03	4.919542e+03	5.663036e+03	2.397717e+06	3.048442e+03	3.663122e+03	5.931006e+03
	min.	9.860790e+02	5.258470e+02	1.209291e+03	8.057080e+02	4.253358e+05	1.241373e+04	3.492038e+03	3.734452e+03	4.143048e+03	2.255279e+06	2.282077e+03	3.157888e+03	5.006455e+03
	max.	1.460314e+03	8.503130e+02	2.769486e+03	1.342671e+03	1.120927e+06	8.158077e+04	6.801850e+03	6.086019e+03	6.824145e+03	2.732663e+06	4.003875e+03	4.148471e+03	6.785785e+03
	std.	9.385167e+01	5.617570e+01	3.953869e+02	1.361076e+02	1.442550e+05	7.382135e+03	8.530517e+02	5.129304e+02	5.648437e+02	1.906202e+05	2.279737e+02	2.399234e+02	2.892536e+02
10D	med.	1.680458e+03	7.436550e+02	2.691724e+03	1.224423e+03	9.026339e+05	2.271864e+04	9.807306e+03	7.366557e+03	9.776943e+03	4.876777e+06	3.644358e+03	5.065917e+03	8.482151e+03
	avg.	1.686060e+03	7.399302e+02	2.546968e+03	1.189198e+03	8.963058e+05	7.054734e+05	9.269635e+03	7.435923e+03	9.810816e+03	4.879721e+06	3.720732e+03	5.149782e+03	8.568786e+03
	min.	1.320334e+03	5.700380e+02	1.242019e+03	8.753670e+02	6.736666e+05	1.442056e+04	6.584946e+03	5.435974e+03	6.837352e+03	4.584884e+06	3.041339e+03	4.464239e+03	7.315172e+03
	max.	2.008135e+03	1.017033e+03	3.227710e+03	1.615639e+03	1.025446e+06	6.753073e+07	1.081728e+04	8.943583e+03	1.134960e+04	5.122731e+06	4.343668e+03	5.847625e+03	9.972161e+03
	std.	1.138808e+02	7.258330e+01	3.678156e+02	1.215940e+02	6.571153e+04	6.750438e+06	1.093257e+03	7.555923e+02	1.052918e+03	1.509802e+05	3.525844e+02	3.392520e+02	5.104703e+02

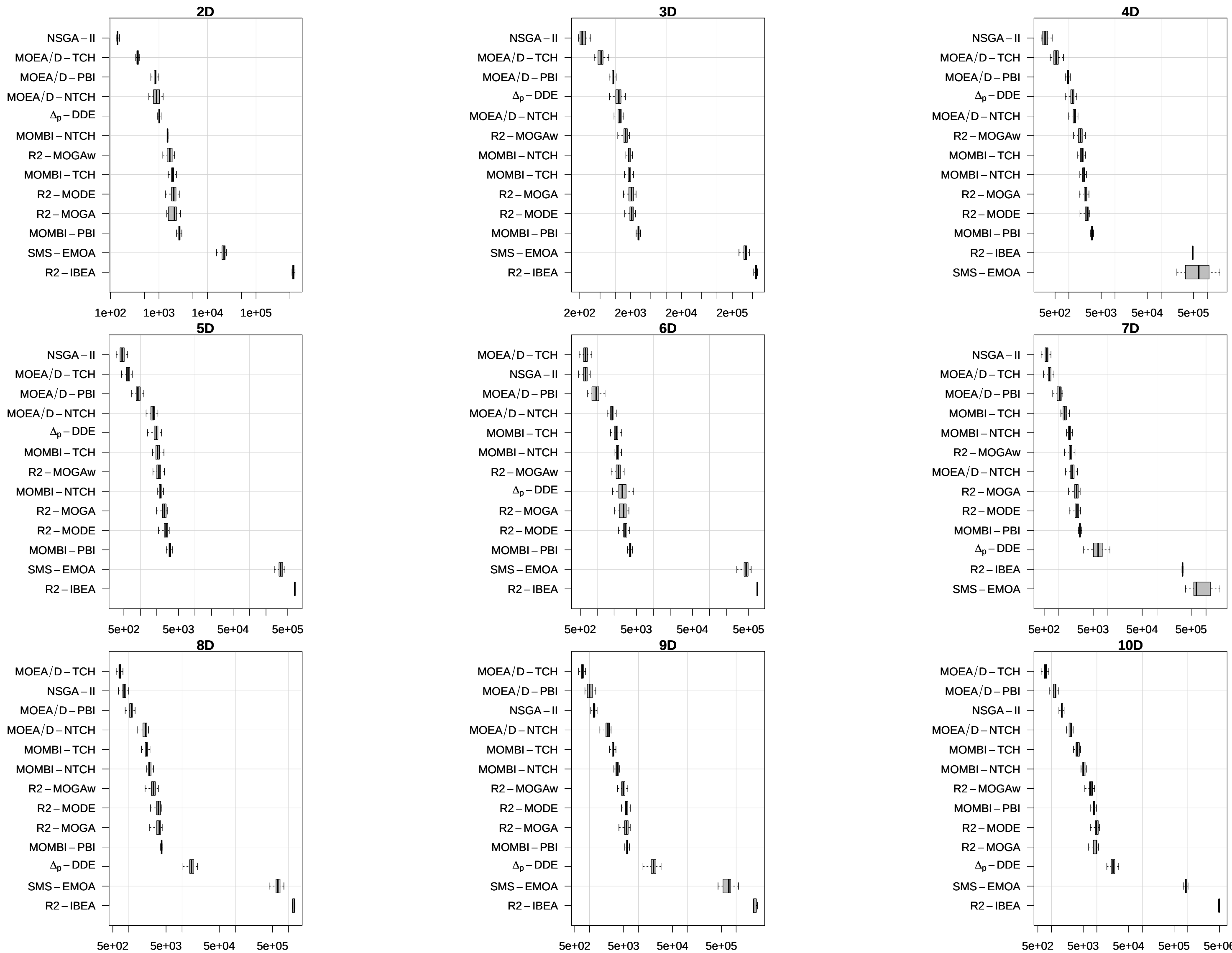


Figure C.3: Box-plot of runtime (in logarithmic scale) for different optimizers on the DTLZ1 test problem.

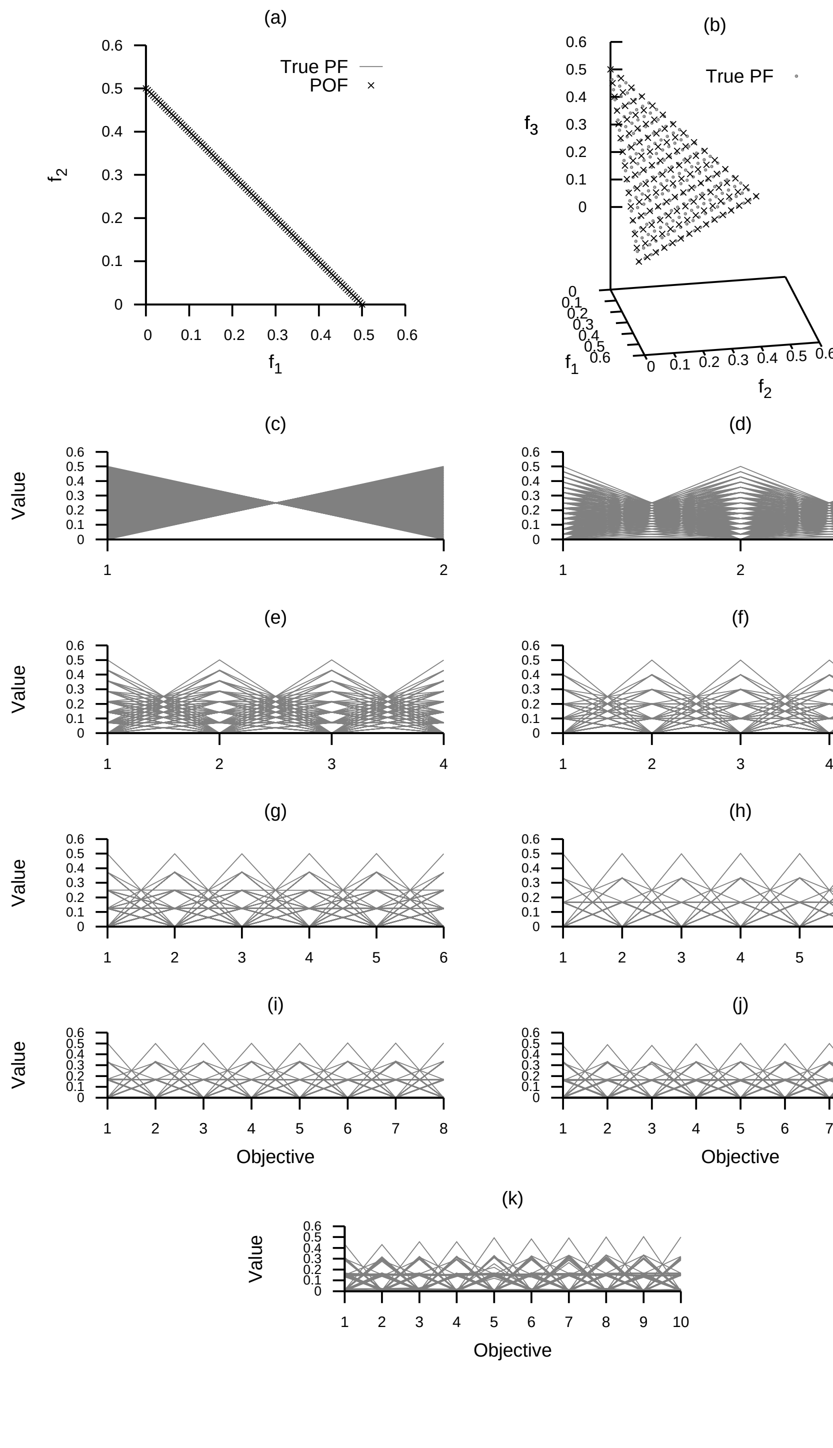


Figure C.4: Plots of the approximations obtained by MOMBI-PBI from 2 to 10 objectives on the DTLZ1 test problem. These plots correspond to the median hypervolume value for 100 independent runs.

Table C.7: Comparison of hypervolume indicator values for different optimizers on the DTLZ2 test problem.

Dim.	Stat.	NSGA-II	MOEA/D TCH	MOEA/D NTCH	MOEA/D PBI	SMS-EMOA	Δ_p-DDE	R2-MOGA	R2-MOGAw	R2-MODE	R2-IBEA	MOMBI TCH	MOMBI NTCH	MOMBI PBI
2D	med.	3.210041e+00	3.210867e+00	3.210768e+00	3.210853e+00	3.211608e+00	3.196187e+00	3.192914e+00	3.193951e+00	3.205840e+00	3.211129e+00	3.210818e+00	3.209403e+00	3.210499e+00
	avg.	3.210061e+00	3.210861e+00	3.210713e+00	3.210840e+00	3.211602e+00	3.196953e+00	3.190293e+00	3.192027e+00	3.205000e+00	3.211126e+00	3.210804e+00	3.209413e+00	3.210489e+00
	min.	3.209661e+00	3.210728e+00	3.209581e+00	3.210688e+00	3.211480e+00	3.155571e+00	3.147819e+00	3.158375e+00	3.175711e+00	3.210942e+00	3.210618e+00	3.208809e+00	3.209876e+00
	max.	3.210394e+00	3.210875e+00	3.210862e+00	3.210874e+00	3.211621e+00	3.210569e+00	3.204530e+00	3.206834e+00	3.207691e+00	3.211267e+00	3.210873e+00	3.209972e+00	3.210841e+00
	std.	1.678848e−04	2.242557e−05	1.883532e−04	3.708775e−05	1.951191e−05	1.274213e−02	1.083726e−02	1.085974e−02	3.957175e−03	6.865480e−05	5.254509e−05	2.074029e−04	1.823577e−04
3D	med.	7.204697e+00	7.383422e+00	6.411807e+00	7.422151e+00	7.431534e+00	7.414369e+00	7.354045e+00	7.326577e+00	7.393166e+00	7.427648e+00	7.383677e+00	7.388695e+00	7.421743e+00
	avg.	7.200007e+00	7.383686e+00	6.442565e+00	7.422149e+00	7.431539e+00	7.392500e+00	7.353170e+00	7.322264e+00	7.392679e+00	7.427598e+00	7.383834e+00	7.388654e+00	7.421723e+00
	min.	7.009836e+00	7.382357e+00	6.408709e+00	7.421998e+00	7.431399e+00	7.315550e+00	7.310730e+00	7.226704e+00	7.382249e+00	7.426628e+00	7.382309e+00	7.383618e+00	7.421200e+00
	max.	7.320675e+00	7.386898e+00	7.294744e+00	7.422207e+00	7.431686e+00	7.417668e+00	7.380991e+00	7.365824e+00	7.401333e+00	7.428481e+00	7.386836e+00	7.394248e+00	7.422053e+00
	std.	5.839806e−02	9.095784e−04	1.403833e−01	3.623787e−05	4.884768e−05	3.233107e−02	1.270901e−02	3.015986e−02	4.243701e−03	4.059323e−04	1.044390e−03	2.242911e−03	1.569161e−04
4D	med.	1.472552e+01	1.542231e+01	1.276200e+01	1.556777e+01	1.189937e+01	1.556396e+01	1.544233e+01	1.540849e+01	1.551662e+01	1.557638e+01	1.542123e+01	1.542185e+01	1.556709e+01
	avg.	1.471135e+01	1.542236e+01	1.283631e+01	1.556775e+01	1.190944e+01	1.556162e+01	1.544101e+01	1.539563e+01	1.551597e+01	1.557642e+01	1.542163e+01	1.542248e+01	1.556708e+01
	min.	1.422657e+01	1.542081e+01	1.255945e+01	1.556748e+01	1.179363e+01	1.544907e+01	1.539562e+01	1.514228e+01	1.549381e+01	1.557358e+01	1.542060e+01	1.542067e+01	1.556669e+01
	max.	1.510316e+01	1.542509e+01	1.506838e+01	1.556797e+01	1.240005e+01	1.556979e+01	1.548211e+01	1.548554e+01	1.553360e+01	1.557900e+01	1.542921e+01	1.542972e+01	1.556741e+01
	std.	1.253025e−01	7.948853e−04	3.559775e−01	1.047063e−04	7.836827e−02	1.238593e−02	1.794148e−02	5.147422e−02	8.098221e−03	1.162881e−03	1.447282e−03	1.742783e−03	1.663787e−04
5D	med.	2.941493e+01	3.153333e+01	2.542240e+01	3.166794e+01	2.430152e+01	3.164526e+01	3.147832e+01	3.144111e+01	3.160021e+01	3.167825e+01	3.153407e+01	3.153469e+01	3.166723e+01
	avg.	2.942620e+01	3.153316e+01	2.547165e+01	3.166796e+01	2.432020e+01	3.164193e+01	3.147656e+01	3.143405e+01	3.159404e+01	3.167816e+01	3.153368e+01	3.153493e+01	3.166721e+01
	min.	2.849557e+01	3.153089e+01	2.532464e+01	3.166701e+01	2.383752e+01	3.149548e+01	3.139351e+01	3.127965e+01	3.136684e+01	3.167360e+01	3.148752e+01	3.153403e+01	3.166591e+01
	max.	3.014224e+01	3.153708e+01	2.859078e+01	3.166860e+01	2.488747e+01	3.167489e+01	3.152941e+01	3.154584e+01	3.162630e+01	3.168224e+01	3.153828e+01	3.154294e+01	3.166800e+01
	std.	2.857214e−01	8.701011e−04	3.241538e−01	2.834108e−04	2.096831e−01	2.538691e−02	2.933922e−02	5.166054e−02	3.665720e−02	1.857753e−03	4.695449e−03	1.140466e−03	3.626224e−04
6D	med.	5.534950e+01	6.282535e+01	5.046712e+01	6.374009e+01	5.305381e+01	6.372645e+01	6.346045e+01	6.346204e+01	6.364523e+01	6.375161e+01	6.308353e+01	6.297146e+01	6.373806e+01
	avg.	5.515635e+01	6.276079e+01	5.069441e+01	6.374007e+01	5.305347e+01	6.370626e+01	6.345828e+01	6.344206e+01	6.357568e+01	6.375149e+01	6.304713e+01	6.297100e+01	6.373805e+01
	min.	5.041419e+01	6.136742e+01	5.045624e+01	6.373933e+01	5.109607e+01	6.231192e+01	6.333743e+01	6.322149e+01	6.233958e+01	6.374371e+01	6.283107e+01	6.285097e+01	6.373635e+01
	max.	5.866762e+01	6.317272e+01	5.736945e+01	6.374083e+01	5.432416e+01	6.375568e+01	6.354051e+01	6.359628e+01	6.368353e+01	6.375564e+01	6.318907e+01	6.317983e+01	6.373926e+01
	std.	1.748605e+00	2.481620e−01	1.057031e+00	3.540214e−04	6.497207e−01	1.433321e−01	4.170817e−02	7.973354e−02	2.460603e−01	2.278054e−03	1.006873e−01	7.957350e−02	5.375246e−04
7D	med.	7.097237e+01	1.221604e+02	1.004964e+02	1.277545e+02	1.003146e+02	1.263361e+02	1.270800e+02	1.269847e+02	1.275699e+02	1.277741e+02	1.218068e+02	1.228784e+02	1.277496e+02
	avg.	6.907730e+01	1.218160e+02	1.011378e+02	1.277545e+02	1.002544e+02	1.262013e+02	1.270752e+02	1.269319e+02	1.273958e+02	1.277739e+02	1.217793e+02	1.225544e+02	1.277494e+02
	min.	2.680668e+01	1.198434e+02	9.987591e+01	1.277538e+02	9.589385e+01	1.240811e+02	1.268092e+02	1.258948e+02	1.252577e+02	1.277628e+02	1.194244e+02	1.203942e+02	1.277455e+02
	max.	9.654712e+01	1.235530e+02	1.153471e+02	1.277548e+02	1.041526e+02	1.271519e+02	1.272888e+02	1.274816e+02	1.276417e+02	1.277822e+02	1.235331e+02	1.231983e+02	1.277521e+02
	std.	1.194559e+01	1.092969e+00	2.665939e+00	1.706577e−04	1.725065e+00	6.213868e−01	9.754212e−02	3.222027e−01	4.775350e−01	3.257479e−03	1.301388e+00	5.964771e−01	1.459325e−03
8D	med.	4.046867e+01	2.436705e+02	2.009920e+02	2.558229e+02	2.182929e+02	2.539939e+02	2.552032e+02	2.552207e+02	2.556410e+02	2.558317e+02	2.439155e+02	2.457868e+02	2.558167e+02
	avg.	4.731003e+01	2.428031e+02	2.016406e+02	2.558228e+02	2.178562e+02	2.539778e+02	2.551761e+02	2.551927e+02	2.554974e+02	2.558317e+02	2.436226e+02	2.452225e+02	2.558164e+02
	min.	1.019203e+01	2.330937e+02	1.986191e+02	2.558201e+02	2.110694e+02	2.521568e+02	2.548047e+02	2.547165e+02	2.533401e+02	2.558213e+02	2.364699e+02	2.413145e+02	2.558089e+02
	max.	1.044286e+02	2.480525e+02	2.275435e+02	2.558243e+02	2.226235e+02	2.552319e+02	2.553962e+02	2.555856e+02	2.557183e+02	2.558409e+02	2.486940e+02	2.491416e+02	2.558200e+02
	std.	2.423026e+01	3.457472e+00	3.801588e+00	8.937217e−04	2.513306e+00	6.518251e−01	1.251305e−01	2.043639e−01	4.230122e−01	3.084503e−03	3.031558e+00	1.193651e+00	2.101846e−03
9D	med.	9.291244e+01	4.878235e+02	4.019926e+02	5.118658e+02	4.624523e+02	5.098186e+02	5.113053e+02	5.113666e+02	5.116906e+02	5.118745e+02	4.881349e+02	4.916219e+02	5.118529e+02
	avg.	9.800724e+01	4.859490e+02	4.026848e+02	5.118654e+02	4.620671e+02	5.098094e+02	5.113004e+02	5.113446e+02	5.115584e+02	5.118738e+02	4.877085e+02	4.910696e+02	5.118520e+02
	min.	2.548915e+01	4.633840e+02	3.989053e+02	5.118548e+02	4.509915e+02	5.073781e+02	5.108276e+02	5.108124e+02	5.054671e+02	5.118642e+02	4.694249e+02	4.865505e+02	5.118332e+02
	max.	2.112485e+02	4.933729e+02	4.513644e+02	5.118727e+02	4.678390e+02	5.111735e+02	5.114953e+02	5.116310e+02	5.117689e+02	5.118806e+02	4.957771e+02	4.977965e+02	5.118630e+02
	std.	4.742846e+01	6.444997e+00	4.987750e+00	3.654564e−03	3.343018e+00	7.599114e−01	1.087475e−01	1.548704e−01	6.714275e−01	4.041045e−03	5.941868e+00	1.743546e+00	5.992567e−03
10D	med.	3.256503e+02	9.573993e+02	8.039549e+02	1.023867e+03	9.616218e+02	1.022179e+03	1.023398e+03	1.023392e+03	1.023824e+03	1.023884e+03	9.785497e+02	9.827967e+02	1.023844e+03
	avg.	3.263275e+02	9.608083e+02	8.063469e+02	1.023866e+03	9.637606e+02	1.022115e+03	1.023368e+03	1.023371e+03	1.023823e+03	1.023882e+03	9.781965e+02	9.815709e+02	1.023843e+03
	min.	1.464623e+02	9.271927e+02	8.008404e+02	1.023811e+03	9.520352e+02	1.020583e+03	1.022759e+03	1.022947e+03	1.023784e+03	1.023843e+03	9.497005e+02	9.729246e+02	1.023779e+03
	max.	6.107166e+02	9.857649e+02	9.076187e+02	1.023897e+03	1.023867e+03	1.023651e+03	1.023540e+03	1.023623e+03	1.023850e+03	1.023905e+03	9.972909e+02	9.949128e+02	1.023878e+03
	std.	8.937217e+01	1.582786e+01	1.460699e+01	1.562131e−02	1.292354e+01	6.017819e−01	1.310344e−01	1.389968e−01	1.352330e−02	1.303181e−02	1.037190e+01	3.926336e+00	1.818843e−02

Table C.8: Comparison of hypervolume indicator values for different optimizers on the DTLZ3 test problem.

Dim.	Stat.	NSGA-II	MOEA/D TCH	MOEA/D NTCH	MOEA/D PBI	SMS-EMOA	Δ_p-DDE	R2-MOGA	R2-MOGAw	R2-MODE	R2-IBEA	MOMBI TCH	MOMBI NTCH	MOMBI PBI
2D	med.	4.793872e+01	4.820301e+01	4.820073e+01	4.818855e+01	4.820349e+01	4.815838e+01	4.810939e+01	4.813783e+01	4.820299e+01	4.820412e+01	4.820382e+01	4.820261e+01	4.819491e+01
	avg.	4.712444e+01	4.820218e+01	4.816813e+01	4.816728e+01	4.820226e+01	4.812366e+01	4.808735e+01	4.812894e+01	4.624848e+01	4.817794e+01	4.820281e+01	4.820219e+01	4.819223e+01
	min.	4.032887e+01	4.817661e+01	4.537692e+01	4.713437e+01	4.818234e+01	4.559775e+01	4.778582e+01	4.787813e+01	2.041612e+01	4.581234e+01	4.819127e+01	4.818335e+01	4.813612e+01
	max.	4.819656e+01	4.820897e+01	4.820735e+01	4.820704e+01	4.821009e+01	4.821055e+01	4.819145e+01	4.819984e+01	4.820720e+01	4.820994e+01	4.820949e+01	4.820809e+01	4.820610e+01
	std.	1.641499e+00	4.947107e−03	2.823876e−01	1.146260e−01	6.000856e−03	2.638653e−01	7.358689e−02	6.025181e−02	4.310835e+00	2.390952e−01	4.236134e−03	4.143913e−03	1.035987e−02
3D	med.	3.213374e+02	3.423777e+02	3.373545e+02	3.423414e+02	3.422325e+02	3.423023e+02	3.420924e+02	3.421533e+02	3.423912e+02	3.424009e+02	3.423709e+02	3.423696e+02	3.423748e+02
	avg.	3.052842e+02	3.423744e+02	3.368608e+02	3.421748e+02	3.401175e+02	3.421819e+02	3.420638e+02	3.420966e+02	3.420053e+02	3.423971e+02	3.423677e+02	3.423247e+02	3.423669e+02
	min.	1.411289e+00	3.423316e+02	2.974115e+02	3.374949e+02	3.024061e+02	3.370165e+02	3.415497e+02	3.412183e+02	3.380363e+02	3.423097e+02	3.422914e+02	3.380038e+02	3.422218e+02
	max.	3.421537e+02	3.423923e+02	3.381744e+02	3.424071e+02	3.424178e+02	3.424147e+02	3.422924e+02	3.423524e+02	3.424005e+02	3.424205e+02	3.423867e+02	3.423871e+02	3.424101e+02
	std.	4.926037e+01	1.065280e−02	3.997101e+00	7.917299e−01	5.826830e+00	5.555253e−01	1.593543e−01	2.185725e−01	1.230164e+00	1.635944e−02	1.565883e−02	4.366046e−01	3.080946e−02
4D	med.	5.349303e+02	2.400411e+03	2.360720e+03	2.399898e+03	1.895105e+03	2.400530e+03	2.399598e+03	2.399739e+03	2.400514e+03	2.400475e+03	2.400392e+03	2.400311e+03	2.400418e+03
	avg.	8.269671e+02	2.400141e+03	2.352650e+03	2.289692e+03	1.594025e+03	2.400462e+03	2.399354e+03	2.399609e+03	2.400297e+03	2.400376e+03	2.400375e+03	2.400203e+03	2.400077e+03
	min.	0.000000e+00	2.390138e+03	2.053514e+03	3.552413e+01	1.376684e+01	2.399290e+03	2.386875e+03	2.397582e+03	2.393233e+03	2.393348e+03	2.399581e+03	2.390753e+03	2.372014e+03
	max.	2.398314e+03	2.400430e+03	2.394949e+03	2.400511e+03	2.396303e+03	2.400560e+03	2.400142e+03	2.400263e+03	2.400533e+03	2.400541e+03	2.400418e+03	2.400423e+03	2.400535e+03
	std.	8.756001e+02	1.521141e+00	4.290223e+01	4.089104e+02	7.515604e+02	1.845585e−01	1.371752e+00	5.899891e−01	1.243124e+00	7.158601e−01	8.521183e−02	9.606724e−01	2.840605e+00
5D	med.	0.000000e+00	1.680644e+04	1.651858e+04	1.678493e+04	4.275212e+03	1.680493e+04	1.680328e+04	1.680401e+04	1.680660e+04	1.680608e+04	1.680645e+04	1.680599e+04	1.680579e+04
	avg.	0.000000e+00	1.676627e+04	1.645687e+04	1.593028e+04	6.268773e+03	1.612477e+04	1.680275e+04	1.680338e+04	1.680641e+04	1.680556e+04	1.680471e+04	1.680385e+04	1.680202e+04
	min.	0.000000e+00	1.336326e+04	1.456703e+04	6.149889e+03	2.341716e+02	6.141943e+02	1.679356e+04	1.678979e+04	1.679428e+04	1.679028e+04	1.676616e+04	1.677309e+04	1.660730e+04
	max.	0.000000e+00	1.680653e+04	1.652714e+04	1.680628e+04	1.599316e+04	1.680645e+04	1.680578e+04	1.680580e+04	1.680662e+04	1.680652e+04	1.680652e+04	1.680617e+04	1.680651e+04
	std.	0.000000e+00	3.442933e+02	2.768688e+02	1.633556e+03	4.792609e+03	2.619019e+03	2.125738e+00	2.357944e+00	1.347691e+00	2.097801e+00	5.950424e+00	7.413873e+00	2.033024e+01
6D	med.	0.000000e+00	1.176217e+05	1.155231e+05	1.152287e+05	3.929994e+04	1.173290e+05	1.176315e+05	1.176355e+05	1.176486e+05	1.176458e+05	1.176438e+05	1.176417e+05	1.176435e+05
	avg.	0.000000e+00	1.175576e+05	1.148434e+05	1.080861e+05	3.784408e+04	1.002488e+05	1.176278e+05	1.176335e+05	1.176412e+05	1.176446e+05	1.176206e+05	1.175901e+05	1.175517e+05
	min.	0.000000e+00	1.168421e+05	9.633443e+04	1.151553e+03	3.913995e+03	8.325118e+03	1.175867e+05	1.175684e+05	1.173995e+05	1.176025e+05	1.169391e+05	1.161017e+05	1.144221e+05
	max.	0.000000e+00	1.176446e+05	1.156564e+05	1.176478e+05	1.058375e+05	1.176487e+05	1.176448e+05	1.176447e+05	1.176487e+05	1.176482e+05	1.176463e+05	1.176430e+05	1.176481e+05
	std.	0.000000e+00	1.266218e+02	2.795486e+03	1.497795e+04	2.923682e+04	3.003094e+04	1.275138e+01	1.108892e+01	3.268303e+01	5.393422e+00	8.758718e+01	2.229975e+02	4.167394e+02
7D	med.	0.000000e+00	8.221654e+05	8.085397e+05	7.082812e+05	2.183438e+05	8.192135e+05	8.232860e+05	8.234046e+05	8.235410e+05	8.235376e+05	8.231668e+05	8.231992e+05	8.235289e+05
	avg.	0.000000e+00	8.213314e+05	8.075167e+05	7.539703e+05	3.469506e+05	8.192135e+05	8.232340e+05	8.233721e+05	8.214184e+05	8.235348e+05	8.225843e+05	8.226020e+05	8.234634e+05
	min.	0.000000e+00	7.681267e+05	7.313920e+05	7.026605e+05	3.696587e+04	8.192135e+05	8.225989e+05	8.229368e+05	6.450634e+05	8.235005e+05	8.050887e+05	7.959358e+05	8.200723e+05
	max.	0.000000e+00	8.233655e+05	8.211467e+05	8.235416e+05	7.641955e+05	8.192135e+05	8.234662e+05	8.235152e+05	8.235426e+05	8.235423e+05	8.233655e+05	8.232577e+05	8.235414e+05
	std.	0.000000e+00	5.935662e+03	7.850325e+03	5.705895e+04	2.708601e+05	0.000000e+00	1.866183e+02	1.060150e+02	1.788382e+04	7.747027e+00	2.209381e+03	2.949610e+03	3.614195e+02
8D	med.	0.000000e+00	5.753074e+06	5.658651e+06	4.951553e+06	1.399787e+06	0.000000e+00	5.764095e+06	5.764350e+06	5.764797e+06	5.764708e+06	5.762570e+06	5.762418e+06	5.764668e+06
	avg.	0.000000e+00	5.753649e+06	5.649521e+06	5.172251e+06	2.028621e+06	0.000000e+00	5.763932e+06	5.764276e+06	5.756736e+06	5.764653e+06	5.758753e+06	5.758742e+06	5.763755e+06
	min.	0.000000e+00	5.661550e+06	5.300696e+06	4.223887e+06	3.163269e+04	0.000000e+00	5.761008e+06	5.763060e+06	5.234734e+06	5.762581e+06	5.635654e+06	5.631064e+06	5.744668e+06
	max.	0.000000e+00	5.763305e+06	5.665631e+06	5.764786e+06	5.464338e+06	0.000000e+00	5.764685e+06	5.764724e+06	5.764801e+06	5.764782e+06	5.763474e+06	5.762808e+06	5.764777e+06
	std.	0.000000e+00	1.425340e+04	4.793877e+04	3.719995e+05	1.923115e+06	0.000000e+00	7.049237e+02	3.384521e+02	5.664051e+04	2.361244e+02	1.485226e+04	1.544533e+04	3.295671e+03
9D	med.	0.000000e+00	4.028434e+07	3.959925e+07	3.463346e+07	3.863997e+06	0.000000e+00	4.035174e+07	4.035195e+07	3.855149e+06	4.035250e+07	4.033735e+07	4.033694e+07	4.035237e+07
	avg.	0.000000e+00	4.025899e+07	3.955216e+07	3.642906e+07	5.542757e+06	0.000000e+00	4.035128e+07	4.035152e+07	8.697830e+06	4.035019e+07	4.022370e+07	4.025918e+07	4.034382e+07
	min.	0.000000e+00	3.965149e+07	3.554514e+07	3.424600e+07	1.417671e+05	0.000000e+00	4.034653e+07	4.034207e+07	2.508055e+03	4.025475e+07	3.662127e+07	3.897948e+07	3.988873e+07
	max.	0.000000e+00	4.034176e+07	4.025057e+07	4.035342e+07	1.108295e+07	0.000000e+00	4.035326e+07	4.035335e+07	3.991437e+07	4.035338e+07	4.034770e+07	4.034056e+07	4.035347e+07
	std.	0.000000e+00	1.422278e+05	4.142387e+05	2.652132e+06	4.656641e+06	0.000000e+00	1.490800e+03	1.722900e+03	1.035168e+07	1.043684e+04	5.059664e+05	2.115673e+05	4.891929e+04
10D	med.	0.000000e+00	2.818423e+08	2.771458e+08	2.423332e+08	0.000000e+00	0.000000e+00	2.824655e+08	2.824677e+08	0.000000e+00	2.824480e+08	2.823592e+08	2.823644e+08	2.824593e+08
	avg.	0.000000e+00	2.815766e+08	2.766366e+08	2.521595e+08	0.000000e+00	0.000000e+00	2.824640e+08	2.824660e+08	0.000000e+00	2.823974e+08	2.814701e+08	2.820167e+08	2.824391e+08
	min.	0.000000e+00	2.773404e+08	2.494457e+08	2.413170e+08	0.000000e+00	0.000000e+00	2.824230e+08	2.824320e+08	0.000000e+00	2.812081e+08	2.545363e+08	2.743435e+08	2.819267e+08
	max.	0.000000e+00	2.847144e+08	2.775625e+08	2.824715e+08	0.000000e+00	0.000000e+00	2.824730e+08	2.824737e+08	0.000000e+00	2.824733e+08	2.824485e+08	2.824587e+08	2.824735e+08
	std.	0.000000e+00	1.222892e+06	3.247150e+06	1.721338e+07	0.000000e+00	0.000000e+00	7.606590e+03	6.706675e+03	0.000000e+00	1.688795e+05	3.285153e+06	1.313959e+06	8.197293e+04

Table C.9: Comparison of hypervolume indicator values for different optimizers on the DTLZ4 test problem.

Dim.	Stat.	NSGA-II	MOEA/D TCH	MOEA/D NTCH	MOEA/D PBI	SMS-EMOA	Δ_p-DDE	R2-MOGA	R2-MOGAw	R2-MODE	R2-IBEA	MOMBI TCH	MOMBI NTCH	MOMBI PBI
2D	med.	3.210149e+00	3.210860e+00	1.999708e+00	3.210851e+00	3.211608e+00	3.201619e+00	3.198168e+00	3.195479e+00	3.201593e+00	3.211102e+00	3.210831e+00	3.209385e+00	3.210601e+00
	avg.	3.210122e+00	2.702303e+00	2.396899e+00	2.739264e+00	2.969280e+00	3.199920e+00	3.195330e+00	3.191610e+00	3.198665e+00	3.090001e+00	3.210826e+00	3.161013e+00	3.198485e+00
	min.	3.209658e+00	2.000000e+00	1.671832e+00	2.000000e+00	2.000000e+00	3.149873e+00	3.162008e+00	3.147484e+00	3.152955e+00	2.000000e+00	3.210672e+00	2.000000e+00	2.000000e+00
	max.	3.210479e+00	3.210875e+00	3.210869e+00	3.210876e+00	3.211620e+00	3.210550e+00	3.204615e+00	3.208349e+00	3.206501e+00	3.211253e+00	3.210875e+00	3.210033e+00	3.210844e+00
	std.	1.853330e−04	6.006445e−01	5.882748e−01	5.927049e−01	4.870815e−01	1.100418e−02	8.890213e−03	1.182593e−02	8.803606e−03	3.651640e−01	3.962216e−05	2.381848e−01	1.210591e−01
3D	med.	7.277431e+00	7.383355e+00	6.409801e+00	6.412261e+00	7.431505e+00	7.413040e+00	7.378895e+00	7.348439e+00	7.387845e+00	7.427891e+00	7.384547e+00	7.391404e+00	7.421943e+00
	avg.	7.173554e+00	6.407321e+00	5.359066e+00	6.400387e+00	6.923149e+00	7.386701e+00	7.358296e+00	7.278159e+00	7.386808e+00	7.309029e+00	7.375186e+00	7.264244e+00	7.421934e+00
	min.	6.419276e+00	4.000000e+00	3.983272e+00	4.000000e+00	4.000000e+00	7.270192e+00	6.393456e+00	6.321684e+00	7.366738e+00	4.000000e+00	6.411823e+00	6.407630e+00	7.421693e+00
	max.	7.389143e+00	7.397226e+00	7.382987e+00	7.422278e+00	7.431620e+00	7.416357e+00	7.394331e+00	7.383941e+00	7.397521e+00	7.428706e+00	7.388772e+00	7.397662e+00	7.422126e+00
	std.	2.827282e−01	1.345481e+00	1.246715e+00	1.318655e+00	8.721746e−01	3.794521e−02	1.381646e−01	2.484379e−01	6.101679e−03	5.236683e−01	9.732370e−02	3.321381e−01	9.456045e−05
4D	med.	1.502053e+01	1.465528e+01	1.274663e+01	1.482292e+01	1.224131e+01	1.556324e+01	1.549269e+01	1.545411e+01	1.551190e+01	1.557719e+01	1.542131e+01	1.542161e+01	1.556774e+01
	avg.	1.498084e+01	1.386932e+01	1.160564e+01	1.375895e+01	1.231870e+01	1.556109e+01	1.546275e+01	1.540696e+01	1.551077e+01	1.554099e+01	1.536473e+01	1.525327e+01	1.556028e+01
	min.	1.447353e+01	8.000000e+00	7.999968e+00	8.000000e+00	1.196911e+01	1.554723e+01	1.474858e+01	1.464395e+01	1.548646e+01	1.485645e+01	1.461214e+01	1.373597e+01	1.482287e+01
	max.	1.519870e+01	1.543181e+01	1.463081e+01	1.556821e+01	1.399390e+01	1.557086e+01	1.551459e+01	1.550128e+01	1.552811e+01	1.557910e+01	1.542950e+01	1.542837e+01	1.556794e+01
	std.	1.749098e−01	2.084705e+00	2.181828e+00	2.491119e+00	2.970671e−01	5.811339e−03	1.444430e−01	1.817467e−01	9.567033e−03	1.577170e−01	1.775341e−01	3.356692e−01	7.448562e−02
5D	med.	3.028211e+01	3.071708e+01	2.535720e+01	3.112059e+01	2.475294e+01	3.165155e+01	3.157379e+01	3.154159e+01	3.158805e+01	3.167999e+01	3.152377e+01	3.151715e+01	3.166883e+01
	avg.	3.025828e+01	2.944722e+01	2.404397e+01	2.947164e+01	2.480758e+01	3.161426e+01	3.152457e+01	3.149010e+01	3.157764e+01	3.164863e+01	3.142473e+01	3.131007e+01	3.166883e+01
	min.	2.936246e+01	1.600000e+01	1.600000e+01	1.600000e+01	2.395866e+01	2.919050e+01	2.952463e+01	3.084945e+01	3.057649e+01	3.116446e+01	3.011126e+01	3.009258e+01	3.166828e+01
	max.	3.084981e+01	3.153403e+01	2.902022e+01	3.166948e+01	2.735323e+01	3.167397e+01	3.160514e+01	3.158061e+01	3.161411e+01	3.168399e+01	3.153602e+01	3.153664e+01	3.166923e+01
	std.	2.653464e−01	2.997700e+00	3.812113e+00	3.361294e+00	5.199166e−01	2.507203e−01	2.396994e−01	1.624858e−01	1.017825e−01	1.227367e−01	2.511012e−01	3.391787e−01	1.961366e−04
6D	med.	5.739084e+01	6.153080e+01	5.032599e+01	6.209191e+01	5.328989e+01	6.370963e+01	6.360839e+01	6.358864e+01	6.361504e+01	6.375348e+01	6.290226e+01	6.297911e+01	6.374110e+01
	avg.	5.721177e+01	5.925554e+01	4.898291e+01	6.046982e+01	5.209563e+01	6.368325e+01	6.356596e+01	6.352445e+01	6.351476e+01	6.372305e+01	6.290487e+01	6.270649e+01	6.374002e+01
	min.	5.236490e+01	3.200000e+01	3.200000e+01	3.200000e+01	4.822283e+01	6.195952e+01	6.199693e+01	6.191744e+01	6.074476e+01	6.336856e+01	6.225694e+01	5.794332e+01	6.371496e+01
	max.	6.062179e+01	6.312065e+01	6.080083e+01	6.374193e+01	5.531635e+01	6.373619e+01	6.365563e+01	6.365519e+01	6.366653e+01	6.375749e+01	6.312389e+01	6.307829e+01	6.374149e+01
	std.	1.758783e+00	7.072223e+00	7.608370e+00	6.741727e+00	2.506627e+00	1.825711e−01	1.944233e−01	2.228918e−01	4.001178e−01	1.034776e−01	1.863104e−01	6.967011e−01	4.224970e−03
7D	med.	3.726546e+01	1.225806e+02	1.000291e+02	1.263956e+02	9.948648e+01	1.270706e+02	1.274621e+02	1.273411e+02	1.274318e+02	1.277739e+02	1.210226e+02	1.221638e+02	1.277534e+02
	avg.	3.960844e+01	1.189622e+02	1.010195e+02	1.218551e+02	9.958144e+01	1.267717e+02	1.271365e+02	1.272256e+02	1.259330e+02	1.276300e+02	1.210806e+02	1.217699e+02	1.277295e+02
	min.	1.268244e+01	6.400000e+01	6.399830e+01	6.400000e+01	7.740984e+01	1.207549e+02	1.170288e+02	1.255314e+02	9.632143e+01	1.265529e+02	1.193990e+02	1.089168e+02	1.273791e+02
	max.	7.404039e+01	1.234743e+02	1.219218e+02	1.277564e+02	1.076220e+02	1.277477e+02	1.275627e+02	1.275759e+02	1.275590e+02	1.277802e+02	1.234812e+02	1.231732e+02	1.277542e+02
	std.	1.366799e+01	1.091385e+01	1.543154e+01	1.441631e+01	5.080616e+00	9.354019e−01	1.136780e+00	3.061829e−01	4.449080e+00	2.540410e−01	9.926296e−01	2.025386e+00	8.329525e−02
8D	med.	3.063069e+01	2.445554e+02	2.002306e+02	2.527888e+02	2.219828e+02	2.553495e+02	2.556052e+02	2.556407e+02	2.552505e+02	2.558344e+02	2.393977e+02	2.439254e+02	2.558094e+02
	avg.	3.508411e+01	2.424790e+02	2.093058e+02	2.467548e+02	2.210593e+02	2.550720e+02	2.555434e+02	2.555261e+02	2.533218e+02	2.557864e+02	2.395485e+02	2.432014e+02	2.557819e+02
	min.	8.595970e+00	1.993214e+02	1.280000e+02	1.280000e+02	1.966670e+02	2.484290e+02	2.549959e+02	2.526925e+02	2.248775e+02	2.550271e+02	2.348143e+02	2.278456e+02	2.552753e+02
	max.	1.111926e+02	2.469791e+02	2.429424e+02	2.558267e+02	2.294180e+02	2.557566e+02	2.556891e+02	2.557270e+02	2.555971e+02	2.558414e+02	2.464383e+02	2.462729e+02	2.558256e+02
	std.	2.012999e+01	7.566822e+00	2.361322e+01	2.096668e+01	5.436193e+00	1.121006e+00	1.526685e−01	3.687652e−01	5.960340e+00	1.182879e−01	2.435533e+00	2.783348e+00	7.755240e−02
9D	med.	8.104468e+01	4.890328e+02	4.453757e+02	5.085828e+02	4.728898e+02	5.108284e+02	5.116154e+02	5.117584e+02	5.105300e+02	5.118498e+02	4.745343e+02	4.850100e+02	5.117262e+02
	avg.	8.866927e+01	4.870021e+02	4.303082e+02	5.043512e+02	4.713741e+02	5.104330e+02	5.115577e+02	5.117049e+02	5.015850e+02	5.117757e+02	4.747901e+02	4.830957e+02	5.116711e+02
	min.	1.365448e+01	4.588531e+02	2.560000e+02	4.533062e+02	4.330758e+02	5.003058e+02	5.109369e+02	5.107048e+02	3.626788e+02	5.106257e+02	4.512683e+02	4.395552e+02	5.111166e+02
	max.	2.378389e+02	4.939586e+02	4.900513e+02	5.118650e+02	5.118513e+02	5.116471e+02	5.117740e+02	5.118092e+02	5.114906e+02	5.118911e+02	4.900741e+02	4.925824e+02	5.118768e+02
	std.	4.858997e+01	6.967582e+00	4.452209e+01	1.143426e+01	1.041781e+01	1.518490e+00	1.777154e−01	1.475158e−01	2.761388e+01	2.110266e−01	6.762019e+00	8.245416e+00	1.752481e−01
10D	med.	5.695858e+02	9.797368e+02	9.175904e+02	1.019569e+03	1.023838e+03	1.023763e+03	1.023772e+03	1.023841e+03	1.023780e+03	1.023920e+03	9.429039e+02	9.703514e+02	1.023911e+03
	avg.	5.764488e+02	9.785347e+02	9.014572e+02	1.017506e+03	1.009283e+03	1.023763e+03	1.023764e+03	1.023830e+03	1.023780e+03	1.023918e+03	9.442357e+02	9.708227e+02	1.023910e+03
	min.	4.291681e+02	9.588862e+02	7.829478e+02	9.361663e+02	9.705047e+02	1.023627e+03	1.023525e+03	1.023587e+03	1.023737e+03	1.023814e+03	9.235966e+02	9.526820e+02	1.023904e+03
	max.	8.044108e+02	9.899378e+02	9.789942e+02	1.023813e+03	1.023904e+03	1.023874e+03	1.023818e+03	1.023864e+03	1.023819e+03	1.023925e+03	9.722101e+02	9.925361e+02	1.023913e+03
	std.	7.907505e+01	6.513080e+00	6.128623e+01	1.117045e+01	2.047838e+01	6.334860e−02	4.902870e−02	4.569562e−02	1.830458e−02	1.495436e−02	1.204980e+01	7.530500e+00	1.812367e−03

Table C.10: Comparison of hypervolume indicator values for different optimizers on the DTLZ5 test problem.

Dim.	Stat.	NSGA-II	MOEA/D TCH	MOEA/D NTCH	MOEA/D PBI	SMS-EMOA	Δ_p-DDE	R2-MOGA	R2-MOGAw	R2-MODE	R2-IBEA	MOMBI TCH	MOMBI NTCH	MOMBI PBI
2D	med.	1.520999e+01	1.521085e+01	1.521056e+01	1.521081e+01	1.521160e+01	1.516908e+01	1.516627e+01	1.516524e+01	1.520575e+01	1.521113e+01	1.521070e+01	1.520922e+01	1.520974e+01
	avg.	1.520999e+01	1.521084e+01	1.521054e+01	1.521077e+01	1.521158e+01	1.516954e+01	1.516244e+01	1.515895e+01	1.520023e+01	1.521113e+01	1.521067e+01	1.520922e+01	1.520971e+01
	min.	1.520920e+01	1.521070e+01	1.520942e+01	1.521032e+01	1.521135e+01	1.504341e+01	1.505132e+01	1.503765e+01	1.504976e+01	1.521092e+01	1.521008e+01	1.520835e+01	1.520828e+01
	max.	1.521038e+01	1.521087e+01	1.521085e+01	1.521087e+01	1.521162e+01	1.521069e+01	1.520421e+01	1.520678e+01	1.520716e+01	1.521125e+01	1.521087e+01	1.520973e+01	1.521085e+01
	std.	1.998792e−04	3.519189e−05	2.165744e−04	1.100434e−04	5.125677e−05	3.812974e−02	2.755311e−02	3.404771e−02	2.402171e−02	6.876960e−05	1.637167e−04	2.541724e−04	5.479907e−04
3D	med.	5.986497e+01	5.984301e+01	5.984079e+01	5.973712e+01	5.987523e+01	5.837975e+01	5.937567e+01	5.926090e+01	5.984303e+01	5.987225e+01	5.984261e+01	5.984361e+01	5.973376e+01
	avg.	5.986478e+01	5.984286e+01	5.984208e+01	5.973648e+01	5.987512e+01	5.809331e+01	5.933040e+01	5.920509e+01	5.970159e+01	5.987216e+01	5.984232e+01	5.984356e+01	5.973364e+01
	min.	5.986062e+01	5.983897e+01	5.983662e+01	5.972660e+01	5.987343e+01	5.557799e+01	5.845189e+01	5.850461e+01	5.826771e+01	5.987068e+01	5.983906e+01	5.983326e+01	5.972221e+01
	max.	5.986745e+01	5.984317e+01	5.985897e+01	5.973802e+01	5.987540e+01	5.924392e+01	5.983309e+01	5.978325e+01	5.985181e+01	5.987284e+01	5.984422e+01	5.985318e+01	5.973823e+01
	std.	1.414396e−03	5.839059e−04	4.073780e−03	1.847066e−03	3.554225e−04	9.973197e−01	2.993668e−01	2.936249e−01	3.069391e−01	4.307426e−04	9.635920e−04	4.171840e−03	3.067849e−03
4D	med.	2.381492e+02	2.388869e+02	2.384235e+02	2.381687e+02	2.385305e+02	2.255927e+02	2.377670e+02	2.367137e+02	2.387219e+02	2.393174e+02	2.388254e+02	2.392075e+02	2.381392e+02
	avg.	2.381213e+02	2.388775e+02	2.381588e+02	2.381681e+02	2.384699e+02	2.265212e+02	2.376615e+02	2.365548e+02	2.387022e+02	2.393029e+02	2.388414e+02	2.391977e+02	2.381417e+02
	min.	2.375795e+02	2.378144e+02	2.364138e+02	2.381181e+02	2.374674e+02	2.157342e+02	2.352136e+02	2.330736e+02	2.368650e+02	2.389353e+02	2.387546e+02	2.387306e+02	2.380980e+02
	max.	2.385441e+02	2.391163e+02	2.390895e+02	2.382059e+02	2.390961e+02	2.357597e+02	2.388759e+02	2.390029e+02	2.393266e+02	2.395660e+02	2.391098e+02	2.395256e+02	2.381848e+02
	std.	2.201643e−01	1.577649e−01	6.918479e−01	1.362063e−02	3.177071e−01	4.885508e+00	6.545821e−01	1.107709e+00	3.267352e−01	1.355417e−01	6.435098e−02	1.587000e−01	1.945659e−02
5D	med.	9.476034e+02	9.454845e+02	9.450054e+02	9.463834e+02	9.117782e+02	8.663383e+02	9.490793e+02	9.458106e+02	9.541181e+02	9.586628e+02	9.457171e+02	9.497518e+02	9.471577e+02
	avg.	9.471790e+02	9.446491e+02	9.450232e+02	9.464136e+02	9.135127e+02	8.695492e+02	9.488449e+02	9.454277e+02	9.539682e+02	9.586229e+02	9.459043e+02	9.497604e+02	9.470515e+02
	min.	9.387188e+02	9.380837e+02	9.374721e+02	9.450435e+02	8.440617e+02	8.088996e+02	9.427675e+02	9.337932e+02	9.490681e+02	9.570900e+02	9.452696e+02	9.451472e+02	9.461580e+02
	max.	9.524148e+02	9.504251e+02	9.519282e+02	9.473757e+02	9.571931e+02	9.339269e+02	9.544960e+02	9.533241e+02	9.569942e+02	9.597234e+02	9.484153e+02	9.533358e+02	9.482842e+02
	std.	2.325150e+00	2.061564e+00	3.023611e+00	3.932463e−01	3.371316e+01	2.526097e+01	2.537586e+00	4.524202e+00	1.623245e+00	5.152468e−01	5.100080e−01	1.790231e+00	4.143114e−01
6D	med.	3.736509e+03	3.752916e+03	3.754141e+03	3.759985e+03	3.295496e+03	3.702004e+03	3.784296e+03	3.782469e+03	3.806124e+03	3.832566e+03	3.752983e+03	3.770434e+03	3.770343e+03
	avg.	3.733598e+03	3.746428e+03	3.756554e+03	3.758882e+03	3.305928e+03	3.599989e+03	3.781805e+03	3.778322e+03	3.806603e+03	3.832533e+03	3.752934e+03	3.770655e+03	3.770362e+03
	min.	3.657452e+03	3.689267e+03	3.707190e+03	3.741322e+03	3.145706e+03	3.180297e+03	3.742643e+03	3.720706e+03	3.793585e+03	3.825664e+03	3.749763e+03	3.749975e+03	3.763968e+03
	max.	3.777694e+03	3.761177e+03	3.793439e+03	3.773582e+03	3.499108e+03	3.780341e+03	3.803193e+03	3.810248e+03	3.819079e+03	3.837932e+03	3.763958e+03	3.798445e+03	3.777086e+03
	std.	2.241531e+01	1.285490e+01	1.530130e+01	6.537366e+00	7.605954e+01	1.743838e+02	1.291392e+01	1.965152e+01	5.437745e+00	2.335078e+00	1.983541e+00	1.120980e+01	2.767205e+00
7D	med.	1.473506e+04	1.491922e+04	1.489100e+04	1.493145e+04	1.164738e+04	1.197908e+04	1.490074e+04	1.499258e+04	1.508832e+04	1.527910e+04	1.492062e+04	1.494703e+04	1.493659e+04
	avg.	1.468789e+04	1.491383e+04	1.488319e+04	1.492906e+04	1.164894e+04	1.194205e+04	1.489161e+04	1.497953e+04	1.508390e+04	1.527346e+04	1.492060e+04	1.495188e+04	1.491422e+04
	min.	1.385514e+04	1.481252e+04	1.474773e+04	1.487578e+04	1.135994e+04	1.038981e+04	1.454943e+04	1.465079e+04	1.497670e+04	1.521042e+04	1.491637e+04	1.482753e+04	1.228801e+04
	max.	1.498080e+04	1.493584e+04	1.501803e+04	1.497385e+04	1.195748e+04	1.313440e+04	1.508953e+04	1.513865e+04	1.517323e+04	1.530555e+04	1.492518e+04	1.506193e+04	1.500021e+04
	std.	2.044862e+02	2.188129e+01	5.296301e+01	1.995835e+01	1.152769e+02	5.653585e+02	1.074693e+02	1.049027e+02	3.554445e+01	2.329846e+01	1.836667e+00	4.883167e+01	2.656117e+02
8D	med.	5.757684e+04	5.954174e+04	5.948930e+04	5.948859e+04	4.972209e+04	4.739453e+04	6.018290e+04	6.023941e+04	6.068430e+04	6.119227e+04	5.978587e+04	5.981579e+04	5.962648e+04
	avg.	5.718632e+04	5.941622e+04	5.947072e+04	5.949243e+04	4.969802e+04	4.774374e+04	6.016760e+04	6.018728e+04	6.067652e+04	6.116308e+04	5.978473e+04	5.983358e+04	5.964046e+04
	min.	4.924415e+04	5.787595e+04	5.839421e+04	5.934399e+04	4.788538e+04	4.371038e+04	5.943169e+04	5.875184e+04	5.995163e+04	6.085203e+04	5.976953e+04	5.947990e+04	5.957604e+04
	max.	5.904669e+04	5.966074e+04	6.010016e+04	5.971451e+04	5.124033e+04	5.853645e+04	6.061351e+04	6.082294e+04	6.088010e+04	6.128942e+04	5.980834e+04	6.038215e+04	5.990995e+04
	std.	1.426610e+03	3.024604e+02	2.704078e+02	7.145102e+01	7.247018e+02	2.726543e+03	2.532080e+02	3.606234e+02	1.339872e+02	1.006032e+02	6.507176e+00	1.747192e+02	5.226583e+01
9D	med.	2.280304e+05	2.376416e+05	2.380527e+05	2.373668e+05	2.407108e+05	1.904470e+05	2.420702e+05	2.417916e+05	2.435827e+05	2.451339e+05	2.398837e+05	2.392650e+05	2.381565e+05
	avg.	2.276276e+05	2.371870e+05	2.380792e+05	2.374341e+05	2.396177e+05	1.924328e+05	2.420771e+05	2.416817e+05	2.434721e+05	2.450340e+05	2.398836e+05	2.392163e+05	2.382060e+05
	min.	2.124206e+05	2.337533e+05	2.284056e+05	2.370485e+05	2.154540e+05	1.697838e+05	2.404748e+05	2.377301e+05	2.405608e+05	2.437626e+05	2.398028e+05	2.359967e+05	2.379745e+05
	max.	2.353520e+05	2.384989e+05	2.408822e+05	2.382538e+05	2.449261e+05	2.346567e+05	2.434202e+05	2.433761e+05	2.445810e+05	2.454298e+05	2.401026e+05	2.410058e+05	2.389368e+05
	std.	4.298323e+03	9.518471e+02	1.522618e+03	2.448896e+02	4.888284e+03	1.260591e+04	7.139773e+02	1.063417e+03	6.074697e+02	3.412859e+02	3.109416e+01	8.033314e+02	1.909000e+02
10D	med.	9.096627e+05	9.487965e+05	9.519180e+05	9.476140e+05	9.770070e+05	7.754288e+05	9.713438e+05	9.693654e+05	9.765996e+05	9.813316e+05	9.613636e+05	9.575009e+05	9.518069e+05
	avg.	9.075072e+05	9.480562e+05	9.504779e+05	9.477215e+05	9.770444e+05	7.847471e+05	9.711617e+05	9.687502e+05	9.764990e+05	9.810410e+05	9.610642e+05	9.575279e+05	9.521429e+05
	min.	8.642879e+05	9.392657e+05	9.304636e+05	9.465297e+05	9.725713e+05	6.951348e+05	9.656803e+05	9.573114e+05	9.735754e+05	9.751037e+05	9.587309e+05	9.469802e+05	9.507105e+05
	max.	9.340905e+05	9.543243e+05	9.615215e+05	9.498431e+05	9.797380e+05	9.321077e+05	9.746233e+05	9.743584e+05	9.782059e+05	9.828043e+05	9.617521e+05	9.647401e+05	9.560001e+05
	std.	1.303655e+04	3.972481e+03	6.303247e+03	6.929602e+02	1.395012e+03	4.865684e+04	2.193105e+03	3.129130e+03	9.817994e+02	1.266587e+03	8.329607e+02	3.208374e+03	1.085091e+03

Table C.11: Comparison of hypervolume indicator values for different optimizers on the DTLZ6 test problem.

Dim.	Stat.	NSGA-II	MOEA/D TCH	MOEA/D NTCH	MOEA/D PBI	SMS-EMOA	Δ_p-DDE	R2-MOGA	R2-MOGAw	R2-MODE	R2-IBEA	MOMBI TCH	MOMBI NTCH	MOMBI PBI
2D	med.	1.201020e+02	1.200317e+02	1.200285e+02	1.200294e+02	1.201051e+02	1.202101e+02	1.199172e+02	1.199622e+02	1.202007e+02	1.202110e+02	1.201022e+02	1.201041e+02	1.201152e+02
	avg.	1.201032e+02	1.200245e+02	1.200340e+02	1.200291e+02	1.200979e+02	1.202101e+02	1.199037e+02	1.199219e+02	1.201576e+02	1.201983e+02	1.201013e+02	1.201021e+02	1.201159e+02
	min.	1.200082e+02	1.198607e+02	1.199063e+02	1.198989e+02	1.200020e+02	1.202093e+02	1.191403e+02	1.195468e+02	1.198362e+02	1.201394e+02	1.199868e+02	1.199514e+02	1.200145e+02
	max.	1.202069e+02	1.201453e+02	1.201816e+02	1.201756e+02	1.202092e+02	1.202105e+02	1.201220e+02	1.201626e+02	1.202089e+02	1.202112e+02	1.202105e+02	1.202079e+02	1.202093e+02
	std.	4.116030e−02	6.205293e−02	5.813202e−02	6.025583e−02	4.614133e−02	2.356806e−04	1.403922e−01	1.408713e−01	7.874017e−02	1.953278e−02	4.511892e−02	4.744676e−02	4.083221e−02
3D	med.	1.316194e+03	1.316780e+03	1.316992e+03	1.316475e+03	1.318068e+03	1.319014e+03	1.313263e+03	1.312541e+03	1.316484e+03	1.319059e+03	1.317911e+03	1.317839e+03	1.313426e+03
	avg.	1.316229e+03	1.316800e+03	1.317035e+03	1.316194e+03	1.318126e+03	1.311892e+03	1.312923e+03	1.312129e+03	1.316260e+03	1.318970e+03	1.317903e+03	1.317828e+03	1.313456e+03
	min.	1.314691e+03	1.315464e+03	1.314953e+03	1.312472e+03	1.317192e+03	1.286301e+03	1.304209e+03	1.301357e+03	1.308908e+03	1.318038e+03	1.316875e+03	1.317078e+03	1.312225e+03
	max.	1.317339e+03	1.318279e+03	1.318605e+03	1.318241e+03	1.319044e+03	1.319039e+03	1.317872e+03	1.316570e+03	1.319034e+03	1.319062e+03	1.318703e+03	1.318646e+03	1.315432e+03
	std.	5.204294e−01	6.015902e−01	6.281976e−01	1.127131e+00	3.854136e−01	9.581243e+00	2.700341e+00	2.936565e+00	2.550088e+00	1.981351e−01	4.157073e−01	3.518532e−01	5.971971e−01
4D	med.	1.389381e+04	1.448145e+04	1.447167e+04	1.446337e+04	1.451076e+04	1.452931e+04	1.447799e+04	1.447177e+04	1.450534e+04	1.452917e+04	1.446003e+04	1.448216e+04	1.442520e+04
	avg.	1.389362e+04	1.448153e+04	1.446330e+04	1.445388e+04	1.444119e+04	1.452900e+04	1.447721e+04	1.446604e+04	1.450271e+04	1.452907e+04	1.445999e+04	1.448211e+04	1.442449e+04
	min.	1.372099e+04	1.446059e+04	1.416101e+04	1.430828e+04	1.355867e+04	1.452397e+04	1.442947e+04	1.436575e+04	1.446963e+04	1.452117e+04	1.443849e+04	1.445944e+04	1.440541e+04
	max.	1.403360e+04	1.450294e+04	1.450888e+04	1.448447e+04	1.452391e+04	1.452990e+04	1.451532e+04	1.450382e+04	1.451469e+04	1.453449e+04	1.447877e+04	1.450095e+04	1.444214e+04
	std.	5.933168e+01	9.387593e+00	4.380031e+01	3.082246e+01	2.213570e+02	9.265028e−01	1.736762e+01	2.658315e+01	9.758201e+00	3.153544e+00	7.547882e+00	9.244019e+00	7.328462e+00
5D	med.	1.337831e+05	1.581836e+05	1.587838e+05	1.587183e+05	1.514058e+05	1.595525e+05	1.593137e+05	1.590080e+05	1.595858e+05	1.597496e+05	1.566880e+05	1.571681e+05	1.586208e+05
	avg.	1.334224e+05	1.581705e+05	1.585674e+05	1.583583e+05	1.528779e+05	1.583533e+05	1.593022e+05	1.589476e+05	1.595276e+05	1.597313e+05	1.565806e+05	1.570593e+05	1.586049e+05
	min.	1.254850e+05	1.573987e+05	1.524036e+05	1.544122e+05	1.468078e+05	1.419973e+05	1.586688e+05	1.577514e+05	1.585021e+05	1.594583e+05	1.543905e+05	1.538183e+05	1.582441e+05
	max.	1.390749e+05	1.589530e+05	1.595374e+05	1.592571e+05	1.597809e+05	1.598039e+05	1.596459e+05	1.593738e+05	1.597127e+05	1.599085e+05	1.574090e+05	1.581485e+05	1.588557e+05
	std.	2.826994e+03	2.803259e+02	9.209921e+02	8.963776e+02	3.995871e+03	4.060067e+03	2.118988e+02	2.784262e+02	1.928830e+02	8.917166e+01	5.250783e+02	7.430711e+02	1.313005e+02
6D	med.	1.210968e+06	1.732205e+06	1.741453e+06	1.743351e+06	1.649778e+06	1.593341e+06	1.752872e+06	1.747305e+06	1.755173e+06	1.755393e+06	1.706020e+06	1.708308e+06	1.741812e+06
	avg.	1.215285e+06	1.732053e+06	1.740146e+06	1.738818e+06	1.658341e+06	1.598306e+06	1.752371e+06	1.747082e+06	1.754668e+06	1.754939e+06	1.703816e+06	1.707671e+06	1.741468e+06
	min.	1.135294e+06	1.722783e+06	1.692235e+06	1.681721e+06	1.602003e+06	1.552980e+06	1.745158e+06	1.739490e+06	1.744455e+06	1.750687e+06	1.677046e+06	1.666214e+06	1.731337e+06
	max.	1.319016e+06	1.741580e+06	1.751741e+06	1.750400e+06	1.755433e+06	1.747672e+06	1.755821e+06	1.751798e+06	1.757194e+06	1.757296e+06	1.721637e+06	1.728577e+06	1.744787e+06
	std.	4.415965e+04	3.792582e+03	8.568077e+03	1.235798e+04	3.987790e+04	3.754295e+04	2.268159e+03	2.208225e+03	1.870405e+03	1.416366e+03	1.010515e+04	9.898368e+03	2.050546e+03
7D	med.	1.168007e+07	1.909011e+07	1.904541e+07	1.920948e+07	1.738885e+07	1.741972e+07	1.920979e+07	1.922346e+07	1.927005e+07	1.925550e+07	1.893897e+07	1.874329e+07	1.921673e+07
	avg.	1.175591e+07	1.907949e+07	1.900003e+07	1.920478e+07	1.743038e+07	1.750680e+07	1.920494e+07	1.921286e+07	1.926325e+07	1.925762e+07	1.890595e+07	1.869999e+07	1.921573e+07
	min.	1.027179e+07	1.890719e+07	1.794389e+07	1.910799e+07	1.714364e+07	1.623055e+07	1.904504e+07	1.905088e+07	1.913544e+07	1.918992e+07	1.843171e+07	1.797550e+07	1.919679e+07
	max.	1.319674e+07	1.919176e+07	1.920868e+07	1.922990e+07	1.822878e+07	1.909564e+07	1.928380e+07	1.927980e+07	1.929755e+07	1.930782e+07	1.905040e+07	1.891535e+07	1.923390e+07
	std.	5.724149e+05	5.460275e+04	2.066098e+05	2.283205e+04	2.138554e+05	6.874916e+05	4.757208e+04	4.537574e+04	2.737833e+04	2.286155e+04	1.244126e+05	1.631819e+05	7.808895e+03
8D	med.	1.198509e+08	2.103817e+08	2.094805e+08	2.107424e+08	1.961256e+08	1.924939e+08	2.118353e+08	2.112575e+08	2.122648e+08	2.119485e+08	2.066746e+08	2.047652e+08	2.102798e+08
	avg.	1.198592e+08	2.102487e+08	2.087880e+08	2.103144e+08	1.965255e+08	1.935899e+08	2.117849e+08	2.111999e+08	2.121740e+08	2.118617e+08	2.066310e+08	2.045215e+08	2.102659e+08
	min.	1.080740e+08	2.090471e+08	2.001751e+08	2.065091e+08	1.912261e+08	1.792699e+08	2.102655e+08	2.104823e+08	2.107168e+08	2.109519e+08	2.033939e+08	1.999911e+08	2.094643e+08
	max.	1.309974e+08	2.112283e+08	2.115651e+08	2.115388e+08	2.125192e+08	2.117038e+08	2.122784e+08	2.118611e+08	2.124617e+08	2.123065e+08	2.095003e+08	2.081151e+08	2.107782e+08
	std.	4.373778e+06	4.762804e+05	2.284221e+06	1.161841e+06	3.385985e+06	9.258986e+06	3.235957e+05	3.271154e+05	3.129269e+05	3.225402e+05	1.225256e+06	1.594303e+06	2.730634e+05
9D	med.	1.322960e+09	2.314140e+09	2.287429e+09	2.306547e+09	2.182929e+09	2.101791e+09	2.328692e+09	2.317241e+09	2.337242e+09	2.328549e+09	2.260390e+09	2.235726e+09	2.291314e+09
	avg.	1.325013e+09	2.313602e+09	2.277142e+09	2.305552e+09	2.184065e+09	2.108699e+09	2.328394e+09	2.316845e+09	2.336784e+09	2.327612e+09	2.259795e+09	2.231659e+09	2.291353e+09
	min.	1.251015e+09	2.296709e+09	2.140775e+09	2.267034e+09	2.147368e+09	2.003190e+09	2.310279e+09	2.303943e+09	2.331822e+09	2.310682e+09	2.237133e+09	2.157961e+09	2.278600e+09
	max.	1.410060e+09	2.326582e+09	2.324737e+09	2.324737e+09	2.278063e+09	2.326528e+09	2.333922e+09	2.325372e+09	2.338853e+09	2.332595e+09	2.281124e+09	2.265226e+09	2.304158e+09
	std.	3.614170e+07	6.314110e+06	3.957895e+07	1.748808e+07	2.446264e+07	8.038022e+07	3.198094e+06	4.111177e+06	1.585609e+06	3.724599e+06	7.551349e+06	2.061718e+07	5.535784e+06
10D	med.	1.506882e+10	2.548168e+10	2.504475e+10	2.553986e+10	2.396791e+10	2.338596e+10	2.558130e+10	2.543025e+10	2.572329e+10	2.555653e+10	2.476779e+10	2.432235e+10	2.492524e+10
	avg.	1.511206e+10	2.546630e+10	2.485604e+10	2.536970e+10	2.402331e+10	2.357800e+10	2.557610e+10	2.542392e+10	2.572129e+10	2.554594e+10	2.475435e+10	2.432592e+10	2.491579e+10
	min.	1.403183e+10	2.522269e+10	2.331958e+10	2.418818e+10	2.367331e+10	2.212991e+10	2.548321e+10	2.520742e+10	2.566648e+10	2.527497e+10	2.440655e+10	2.353210e+10	2.466304e+10
	max.	1.628967e+10	2.557659e+10	2.554893e+10	2.557207e+10	2.475394e+10	2.570769e+10	2.562975e+10	2.551541e+10	2.573830e+10	2.560977e+10	2.491899e+10	2.475823e+10	2.510424e+10
	std.	4.081312e+08	6.705519e+07	5.824512e+08	2.827193e+08	2.189607e+08	1.003616e+09	2.616204e+07	5.253210e+07	1.190040e+07	5.728045e+07	9.075315e+07	2.122051e+08	8.095296e+07

Table C.12: Comparison of hypervolume indicator values for different optimizers on the DTLZ7 test problem.

Dim.	Stat.	NSGA-II	MOEA/D TCH	MOEA/D NTCH	MOEA/D PBI	SMS-EMOA	Δ_p-DDE	R2-MOGA	R2-MOGAw	R2-MODE	R2-IBEA	MOMBI TCH	MOMBI NTCH	MOMBI PBI
2D	med.	1.772480e + 01	1.737751e + 01	1.737719e + 01	1.771931e + 01	1.772581e + 01	1.761423e + 01	1.738405e + 01	1.728211e + 01	1.770631e + 01	1.772491e + 01	1.772192e + 01	1.772291e + 01	1.771474e + 01
	avg.	1.772479e + 01	1.750186e + 01	1.744287e + 01	1.766095e + 01	1.772574e + 01	1.760818e + 01	1.728308e + 01	1.710696e + 01	1.758817e + 01	1.772492e + 01	1.772198e + 01	1.772275e + 01	1.771466e + 01
	min.	1.772444e + 01	1.737699e + 01	1.737639e + 01	1.737531e + 01	1.772478e + 01	1.701471e + 01	1.588911e + 01	1.505963e + 01	1.648050e + 01	1.772430e + 01	1.771780e + 01	1.772070e + 01	1.770228e + 01
	max.	1.772510e + 01	1.772406e + 01	1.772460e + 01	1.772181e + 01	1.772584e + 01	1.772257e + 01	1.770816e + 01	1.769911e + 01	1.771637e + 01	1.772521e + 01	1.772410e + 01	1.772409e + 01	1.772230e + 01
	std.	1.418413e − 04	1.667318e − 01	1.365132e − 01	1.294566e − 01	1.812639e − 04	1.171247e − 01	3.712635e − 01	5.811945e − 01	2.570454e − 01	1.634515e − 04	1.413655e − 03	8.078977e − 04	4.464758e − 03
3D	med.	1.625727e + 01	1.611884e + 01	1.574321e + 01	1.624352e + 01	1.616077e + 01	1.568592e + 01	1.522768e + 01	1.510171e + 01	1.619958e + 01	1.636467e + 01	1.611713e + 01	1.630053e + 01	1.624761e + 01
	avg.	1.621093e + 01	1.603450e + 01	1.586085e + 01	1.623135e + 01	1.616356e + 01	1.562234e + 01	1.512256e + 01	1.466061e + 01	1.604402e + 01	1.636393e + 01	1.611807e + 01	1.628683e + 01	1.624675e + 01
	min.	1.585559e + 01	1.574287e + 01	1.572072e + 01	1.574385e + 01	1.590868e + 01	1.400619e + 01	1.255174e + 01	9.719387e + 00	1.347709e + 01	1.633119e + 01	1.611037e + 01	1.605853e + 01	1.622681e + 01
	max.	1.630807e + 01	1.612751e + 01	1.630188e + 01	1.625842e + 01	1.637037e + 01	1.630684e + 01	1.616368e + 01	1.612044e + 01	1.630300e + 01	1.637204e + 01	1.618206e + 01	1.632061e + 01	1.625708e + 01
	std.	1.016939e − 01	1.204839e − 01	1.731456e − 01	7.079816e − 02	1.155159e − 01	4.323448e − 01	6.824288e − 01	1.406949e + 00	4.312167e − 01	5.900564e − 03	7.286148e − 03	5.769655e − 02	5.312461e − 03
4D	med.	1.423275e + 01	1.405896e + 01	1.408219e + 01	1.403204e + 01	1.451449e + 01	1.220876e + 01	1.214944e + 01	1.187276e + 01	1.440554e + 01	1.480566e + 01	1.383563e + 01	1.449625e + 01	1.410244e + 01
	avg.	1.417787e + 01	1.401840e + 01	1.417979e + 01	1.400633e + 01	1.431565e + 01	1.224598e + 01	1.209790e + 01	1.144165e + 01	1.424559e + 01	1.479669e + 01	1.384247e + 01	1.449576e + 01	1.409655e + 01
	min.	1.334970e + 01	1.383230e + 01	1.403648e + 01	1.343959e + 01	1.015739e + 01	8.335459e + 00	8.878037e + 00	4.785225e + 00	1.155970e + 01	1.435243e + 01	1.382334e + 01	1.433534e + 01	1.399082e + 01
	max.	1.442015e + 01	1.419204e + 01	1.452598e + 01	1.414910e + 01	1.469439e + 01	1.402565e + 01	1.406742e + 01	1.418329e + 01	1.461041e + 01	1.484578e + 01	1.406102e + 01	1.455359e + 01	1.413556e + 01
	std.	1.959550e − 01	1.402125e − 01	1.481472e − 01	1.110643e − 01	7.982446e − 01	1.159265e + 00	1.042412e + 00	2.162405e + 00	5.284867e − 01	5.235353e − 02	3.373519e − 02	3.097528e − 02	2.672768e − 02
5D	med.	1.128330e + 01	1.207228e + 01	1.239050e + 01	6.632180e + 00	1.254955e + 01	9.530219e + 00	9.327853e + 00	9.002057e + 00	1.204837e + 01	1.301004e + 01	1.195363e + 01	1.246622e + 01	7.930343e + 00
	avg.	1.127591e + 01	1.215301e + 01	1.245269e + 01	6.582571e + 00	1.254674e + 01	9.112810e + 00	9.271454e + 00	8.485852e + 00	1.159792e + 01	1.300513e + 01	1.199014e + 01	1.249499e + 01	6.397375e + 00
	min.	9.938904e + 00	1.195956e + 01	1.233506e + 01	1.942670e + 00	1.237415e + 01	6.004364e + 00	4.295885e + 00	1.752453e + 00	5.004023e + 00	1.283646e + 01	1.193280e + 01	1.238484e + 01	3.957258e − 01
	max.	1.218175e + 01	1.243445e + 01	1.271663e + 01	1.214119e + 01	1.266316e + 01	1.212565e + 01	1.136384e + 01	1.202672e + 01	1.265419e + 01	1.312772e + 01	1.222778e + 01	1.267267e + 01	1.147891e + 01
	std.	4.292879e − 01	1.555332e − 01	1.098797e − 01	1.421676e + 00	5.516576e − 02	1.681988e + 00	9.929234e − 01	2.485358e + 00	1.318756e + 00	6.379519e − 02	6.862621e − 02	7.419639e − 02	3.662227e + 00
6D	med.	7.201025e + 00	1.001897e + 01	1.071389e + 01	3.010195e − 01	1.049732e + 01	8.707716e + 00	6.587967e + 00	6.483835e + 00	9.483471e + 00	1.087549e + 01	6.026100e + 00	9.679229e + 00	1.689105e − 01
	avg.	7.198799e + 00	9.709019e + 00	1.070585e + 01	7.559983e − 01	1.049120e + 01	8.295555e + 00	6.524449e + 00	6.328746e + 00	8.973341e + 00	1.081639e + 01	7.514874e + 00	9.013127e + 00	3.009163e − 01
	min.	5.398210e + 00	5.944880e + 00	1.006647e + 01	1.680153e − 02	1.020933e + 01	2.221281e + 00	2.322481e + 00	4.254753e − 01	1.761578e + 00	9.721319e + 00	5.884775e + 00	7.209575e + 00	1.101620e − 02
	max.	8.373582e + 00	1.066714e + 01	1.087544e + 01	4.611917e + 00	1.070468e + 01	9.000100e + 00	8.727890e + 00	1.021762e + 01	1.058413e + 01	1.129017e + 01	1.040089e + 01	1.082052e + 01	8.685237e + 00
	std.	4.721324e − 01	1.157947e + 00	1.119831e − 01	1.106340e + 00	1.116773e − 01	1.142310e + 00	1.126137e + 00	2.519902e + 00	1.452061e + 00	3.095007e − 01	1.743132e + 00	1.192963e + 00	8.880978e − 01
7D	med.	3.419147e + 00	5.386171e + 00	8.744941e + 00	3.270779e − 02	8.255152e + 00	4.921470e + 00	3.521101e + 00	2.171392e + 00	4.634357e + 00	3.943842e + 00	1.297963e + 00	5.000797e + 00	5.457545e − 03
	avg.	3.492283e + 00	5.079091e + 00	8.715724e + 00	3.418151e − 01	7.660584e + 00	4.815163e + 00	3.224791e + 00	2.646788e + 00	4.675548e + 00	4.464070e + 00	1.518496e + 00	4.949603e + 00	1.160390e − 02
	min.	2.342515e + 00	4.903421e − 01	8.001329e + 00	1.734397e − 03	9.536540e − 03	1.919996e + 00	4.283396e − 01	4.177557e − 02	4.454769e − 01	2.956974e + 00	4.805222e − 01	2.261152e + 00	4.624768e − 04
	max.	5.141579e + 00	8.694043e + 00	8.892895e + 00	3.355103e + 00	8.822642e + 00	6.826996e + 00	5.238408e + 00	7.291774e + 00	8.121388e + 00	8.932015e + 00	5.412484e + 00	8.485949e + 00	2.014949e − 01
	std.	5.137960e − 01	2.323108e + 00	1.171019e − 01	7.387445e − 01	2.206066e + 00	1.032515e + 00	1.227144e + 00	2.079665e + 00	1.961746e + 00	1.411136e + 00	1.216228e + 00	1.640278e + 00	2.287819e − 02
8D	med.	5.449157e − 01	2.009202e + 00	6.861234e + 00	5.760527e − 03	5.982937e + 00	1.000560e + 00	2.895890e + 00	2.772100e + 00	3.902094e + 00	1.943097e + 00	1.212241e − 01	2.681736e + 00	7.834286e − 04
	avg.	5.408725e − 01	2.639309e + 00	6.841700e + 00	8.569387e − 02	5.970082e + 00	1.137232e + 00	2.766674e + 00	2.553062e + 00	3.707429e + 00	2.968247e + 00	3.207260e − 01	2.894473e + 00	1.156260e − 03
	min.	1.546198e − 01	1.132322e − 01	6.459977e + 00	2.953331e − 04	5.113759e + 00	1.509022e − 02	2.548662e − 01	3.858361e − 02	2.795593e − 01	1.874660e + 00	1.114375e − 01	1.182214e + 00	8.665254e − 05
	max.	1.099832e + 00	6.861088e + 00	7.082651e + 00	9.749990e − 01	6.721674e + 00	3.374391e + 00	3.901195e + 00	6.106034e + 00	6.060959e + 00	7.419353e + 00	1.993311e + 00	6.812438e + 00	4.352052e − 03
	std.	1.971527e − 01	1.825037e + 00	1.269969e − 01	1.908364e − 01	3.488228e − 01	8.292815e − 01	7.788717e − 01	1.628632e + 00	1.469107e + 00	1.576627e + 00	3.075117e − 01	1.284603e + 00	1.040938e − 03
9D	med.	4.573806e − 03	1.319000e + 00	4.906740e + 00	4.076489e − 03	3.130287e + 00	2.875505e − 01	2.163168e + 00	1.391094e + 00	3.177093e + 00	1.398052e + 00	2.660244e − 02	1.683408e + 00	1.196427e − 04
	avg.	8.350265e − 03	1.587307e + 00	4.902256e + 00	3.022224e − 02	3.099550e + 00	3.234025e − 01	2.149656e + 00	1.473555e + 00	3.022422e + 00	1.627963e + 00	6.174390e − 02	2.046943e + 00	1.156431e − 04
	min.	3.866863e − 04	2.597906e − 02	4.534856e + 00	5.539845e − 05	2.048496e + 00	1.454994e − 02	6.716342e − 01	2.273063e − 02	8.084795e − 01	1.351538e + 00	2.265383e − 02	6.732394e − 01	1.761664e − 05
	max.	5.837259e − 02	4.521861e + 00	5.200088e + 00	2.425406e − 01	3.693270e + 00	5.875595e − 01	3.085832e + 00	3.434968e + 00	4.175770e + 00	4.566778e + 00	5.938563e − 01	5.258115e + 00	6.197188e − 04
	std.	8.977934e − 03	1.271320e + 00	1.627951e − 01	5.959820e − 02	3.378674e − 01	2.400987e − 01	4.680234e − 01	1.014844e + 00	7.508740e − 01	5.543930e − 01	7.605415e − 02	1.121133e + 00	8.070427e − 05
10D	med.	5.249787e + 00	3.903390e − 01	2.893759e + 00	2.857973e − 03	9.623705e − 01	1.216161e + 01	1.443151e + 00	6.934844e − 01	1.585825e + 00	1.002269e + 00	5.590796e − 03	9.478212e − 01	1.786547e − 05
	avg.	5.455100e + 00	7.091379e − 01	2.877315e + 00	1.188268e − 02	1.041105e + 00	1.220766e + 01	1.409198e + 00	8.144383e − 01	1.572889e + 00	1.063874e + 00	8.637017e − 03	1.045174e + 00	1.488646e − 05
	min.	2.614008e + 00	5.438431e − 03	2.248070e + 00	1.134181e − 05	4.539230e − 02	5.765086e + 00	2.811418e − 01	1.449959e − 02	1.866188e − 01	9.574546e − 01	4.489874e − 03	3.297817e − 01	1.965790e − 06
	max.	9.633890e + 00	2.919172e + 00	3.337563e + 00	1.453774e − 01	2.053183e + 00	1.944170e + 01	2.187026e + 00	3.068195e + 00	2.004756e + 00	1.823787e + 00	5.655958e − 02	2.582186e + 00	2.649139e − 05
	std.	1.606683e + 00	6.755171e − 01	2.509435e − 01	2.371467e − 02	4.555894e − 01	2.530226e + 00	3.256343e − 01	5.947873e − 01	2.845953e − 01	1.594379e − 01	9.093379e − 03	4.671878e − 01	6.120255e − 06

Table C.13: Comparison of hypervolume indicator values for different optimizers on the WFG1 test problem.

Dim.	Stat.	NSGA-II	MOEA/D TCH	MOEA/D NTCH	MOEA/D PBI	SMS-EMOA	Δ_p-DDE	R2-MOGA	R2-MOGAw	R2-MODE	R2-IBEA	MOMBI TCH	MOMBI NTCH	MOMBI PBI
2D	med.	5.462359e+00	4.914358e+00	5.364344e+00	4.299120e+00	5.216719e+00	5.875994e+00	5.846619e+00	5.912147e+00	5.676851e+00	8.532078e+00	5.638118e+00	5.842622e+00	4.906127e+00
	avg.	5.613198e+00	4.994187e+00	5.449888e+00	4.295479e+00	5.218911e+00	5.878426e+00	5.957711e+00	6.005061e+00	5.684949e+00	8.698179e+00	5.626208e+00	6.000237e+00	4.968546e+00
	min.	4.871588e+00	3.265890e+00	3.971556e+00	3.617274e+00	4.506204e+00	5.769845e+00	5.067192e+00	5.067505e+00	5.537496e+00	7.754294e+00	4.793036e+00	5.174677e+00	4.398853e+00
	max.	7.120354e+00	6.259757e+00	6.538528e+00	4.817146e+00	7.409844e+00	5.983787e+00	7.188923e+00	7.484200e+00	5.849240e+00	1.038463e+01	6.392600e+00	7.452735e+00	6.386459e+00
	std.	5.029624e−01	5.607255e−01	5.744606e−01	2.055168e−01	3.931786e−01	4.617552e−02	5.528027e−01	5.470447e−01	5.805579e−02	7.636584e−01	3.025477e−01	5.286265e−01	2.928271e−01
3D	med.	3.568445e+01	5.298790e+01	4.864920e+01	4.969971e+01	6.018661e+01	4.763078e+01	5.459438e+01	5.695421e+01	4.910925e+01	7.553449e+01	5.391138e+01	5.449829e+01	4.917298e+01
	avg.	3.573018e+01	5.227857e+01	4.853772e+01	4.847948e+01	5.971890e+01	4.766870e+01	5.463634e+01	5.702944e+01	4.909904e+01	7.665978e+01	5.385996e+01	5.439026e+01	4.850851e+01
	min.	3.355509e+01	4.756529e+01	3.959740e+01	4.296551e+01	5.274862e+01	4.687834e+01	5.281628e+01	5.474343e+01	4.793777e+01	6.881641e+01	5.055970e+01	5.016687e+01	4.257459e+01
	max.	3.878649e+01	5.469985e+01	5.354379e+01	5.176109e+01	6.357477e+01	4.847397e+01	5.775135e+01	5.903375e+01	5.035301e+01	8.392603e+01	5.578139e+01	5.693552e+01	5.126336e+01
	std.	9.222034e−01	1.915141e+00	2.910133e+00	2.715035e+00	2.338678e+00	3.185352e−01	9.476713e−01	9.201671e−01	4.561858e−01	4.231603e+00	7.929417e−01	1.243932e+00	2.191201e+00
4D	med.	2.999472e+02	4.363328e+02	3.752584e+02	3.773955e+02	3.843418e+02	3.936088e+02	4.547109e+02	4.676028e+02	4.335961e+02	5.972767e+02	4.335852e+02	4.328992e+02	3.758139e+02
	avg.	3.000457e+02	4.325936e+02	3.743511e+02	3.872948e+02	3.852523e+02	3.951974e+02	4.549730e+02	4.672755e+02	4.335653e+02	5.997024e+02	4.324732e+02	4.322864e+02	3.840830e+02
	min.	2.822502e+02	4.008668e+02	3.178595e+02	3.559593e+02	3.675130e+02	3.826978e+02	4.330471e+02	4.447273e+02	4.261006e+02	5.663855e+02	4.136064e+02	4.106206e+02	3.574447e+02
	max.	3.177772e+02	4.545648e+02	4.322106e+02	4.299096e+02	4.284461e+02	4.121265e+02	4.757274e+02	4.841930e+02	4.435653e+02	6.722141e+02	4.449113e+02	4.460360e+02	4.140905e+02
	std.	6.853279e+00	1.300987e+01	2.340402e+01	2.169376e+01	1.103903e+01	5.704333e+00	8.068783e+00	7.401526e+00	3.409073e+00	2.039213e+01	6.627654e+00	7.281924e+00	1.843906e+01
5D	med.	3.165284e+03	4.205915e+03	3.449449e+03	3.785514e+03	3.637982e+03	3.997044e+03	4.439048e+03	4.551241e+03	4.420813e+03	5.908312e+03	4.346138e+03	4.283159e+03	3.713781e+03
	avg.	3.164572e+03	4.239323e+03	3.442911e+03	3.837411e+03	3.644455e+03	3.992746e+03	4.447962e+03	4.556431e+03	4.424050e+03	5.920511e+03	4.350517e+03	4.299735e+03	3.754181e+03
	min.	3.014939e+03	4.022654e+03	2.783251e+03	3.584182e+03	3.391319e+03	3.822199e+03	4.322882e+03	4.377570e+03	4.363041e+03	5.517745e+03	4.170580e+03	4.167911e+03	3.594959e+03
	max.	3.295570e+03	4.515805e+03	4.152484e+03	4.493660e+03	3.888046e+03	4.126977e+03	4.618256e+03	4.736161e+03	4.498066e+03	6.495744e+03	4.582562e+03	4.491752e+03	4.179526e+03
	std.	6.297992e+01	1.184932e+02	2.327068e+02	1.836751e+02	8.322119e+01	5.933607e+01	7.429734e+01	9.237423e+01	3.024078e+01	1.638093e+02	6.802183e+01	7.933194e+01	1.291893e+02
6D	med.	3.961896e+04	5.058502e+04	4.075742e+04	4.537424e+04	4.178292e+04	4.745718e+04	5.276427e+04	5.397905e+04	5.287953e+04	7.143837e+04	5.261463e+04	5.178997e+04	4.549634e+04
	avg.	3.962700e+04	5.077014e+04	4.079408e+04	4.628078e+04	4.170160e+04	4.749803e+04	5.283112e+04	5.414936e+04	5.291284e+04	7.153553e+04	5.259595e+04	5.186736e+04	4.570715e+04
	min.	3.808494e+04	4.837625e+04	3.428403e+04	4.340534e+04	3.833528e+04	4.640198e+04	5.084747e+04	5.253975e+04	5.191570e+04	6.672165e+04	4.925144e+04	5.047884e+04	4.337861e+04
	max.	4.099992e+04	5.370963e+04	4.701061e+04	5.194246e+04	4.452259e+04	4.937755e+04	5.537359e+04	5.700767e+04	5.356893e+04	7.811093e+04	5.500136e+04	5.445637e+04	5.040109e+04
	std.	6.426644e+02	9.744763e+02	2.550162e+03	2.280993e+03	1.174205e+03	5.115224e+02	9.125991e+02	8.821086e+02	2.894650e+02	2.130271e+03	9.485106e+02	6.750439e+02	1.126724e+03
7D	med.	5.675085e+05	7.103932e+05	5.507385e+05	6.592825e+05	5.947915e+05	6.645683e+05	7.317450e+05	7.710217e+05	7.747603e+05	1.138909e+06	7.601295e+05	7.547361e+05	6.828833e+05
	avg.	5.671659e+05	7.100479e+05	5.483226e+05	6.640954e+05	5.932660e+05	6.651603e+05	7.340535e+05	7.711226e+05	7.750353e+05	1.137462e+06	7.587543e+05	7.553591e+05	6.826636e+05
	min.	5.345314e+05	6.827213e+05	4.355632e+05	6.246522e+05	5.279304e+05	6.542456e+05	7.120322e+05	7.355933e+05	7.596985e+05	1.011569e+06	7.097780e+05	7.325436e+05	6.434942e+05
	max.	5.836994e+05	7.475913e+05	6.510853e+05	7.375575e+05	6.505399e+05	6.797777e+05	7.671301e+05	8.056107e+05	7.948739e+05	1.222339e+06	7.872074e+05	7.887441e+05	7.536961e+05
	std.	8.653643e+03	1.230891e+04	4.182843e+04	2.390291e+04	2.030456e+04	5.905857e+03	1.096246e+04	1.273553e+04	6.319036e+03	4.066803e+04	1.441700e+04	1.079752e+04	1.887342e+04
8D	med.	9.328351e+06	1.136470e+07	8.575436e+06	1.065207e+07	8.692666e+06	1.061006e+07	1.183472e+07	1.210138e+07	1.172177e+07	1.691765e+07	1.190240e+07	1.191409e+07	1.135197e+07
	avg.	9.339379e+06	1.133524e+07	8.693511e+06	1.070343e+07	8.705381e+06	1.060913e+07	1.182022e+07	1.213613e+07	1.172530e+07	1.691214e+07	1.191395e+07	1.191702e+07	1.134547e+07
	min.	9.104162e+06	1.083173e+07	6.739908e+06	1.010184e+07	8.097936e+06	1.044831e+07	1.146003e+07	1.175905e+07	1.156896e+07	1.544364e+07	1.142539e+07	1.160367e+07	1.088232e+07
	max.	9.649505e+06	1.181545e+07	1.048732e+07	1.185216e+07	9.354137e+06	1.070902e+07	1.212068e+07	1.282519e+07	1.194497e+07	1.791833e+07	1.229190e+07	1.231945e+07	1.189698e+07
	std.	9.730284e+04	2.058054e+05	6.597119e+05	3.352670e+05	2.702971e+05	4.808386e+04	1.376781e+05	1.779200e+05	6.899932e+04	4.272451e+05	1.928349e+05	1.354452e+05	1.888264e+05
9D	med.	1.708704e+08	2.009463e+08	1.511786e+08	1.909083e+08	1.504182e+08	1.911000e+08	2.131923e+08	2.166479e+08	2.049415e+08	2.691906e+08	2.110870e+08	2.118505e+08	2.050601e+08
	avg.	1.710400e+08	2.009571e+08	1.525088e+08	1.912840e+08	1.504794e+08	1.912819e+08	2.132641e+08	2.168482e+08	2.050181e+08	2.694135e+08	2.112527e+08	2.117147e+08	2.050124e+08
	min.	1.680926e+08	1.892866e+08	1.298587e+08	1.835622e+08	1.363577e+08	1.901592e+08	2.086256e+08	2.120742e+08	2.035420e+08	2.553186e+08	2.051766e+08	2.052986e+08	1.995332e+08
	max.	1.754480e+08	2.112792e+08	1.835066e+08	2.091752e+08	1.614670e+08	1.936965e+08	2.218463e+08	2.221525e+08	2.078647e+08	2.832295e+08	2.163483e+08	2.227167e+08	2.101744e+08
	std.	1.464659e+06	3.571941e+06	1.073714e+07	3.856771e+06	4.275982e+06	7.473478e+05	2.098197e+06	1.973221e+06	8.098268e+05	5.995060e+06	2.354342e+06	2.214950e+06	2.467371e+06
10D	med.	3.449011e+09	4.000843e+09	3.043455e+09	3.801957e+09	2.836797e+09	3.820934e+09	4.259319e+09	4.295204e+09	4.025032e+09	4.722387e+09	4.163104e+09	4.173288e+09	4.058654e+09
	avg.	3.447654e+09	4.006121e+09	3.041157e+09	3.804224e+09	2.848816e+09	3.824122e+09	4.261389e+09	4.298027e+09	4.026265e+09	4.726504e+09	4.167185e+09	4.176136e+09	4.062747e+09
	min.	3.386511e+09	3.830700e+09	2.558320e+09	3.674781e+09	2.536466e+09	3.798558e+09	4.191595e+09	4.212989e+09	4.000947e+09	4.559714e+09	4.048004e+09	4.104300e+09	3.940567e+09
	max.	3.517417e+09	4.169867e+09	3.572187e+09	4.149556e+09	3.143816e+09	3.910851e+09	4.333448e+09	4.379027e+09	4.059071e+09	4.925270e+09	4.339987e+09	4.412840e+09	4.191077e+09
	std.	2.552201e+07	6.444808e+07	1.629448e+08	6.856123e+07	9.937826e+07	1.559904e+07	2.840457e+07	3.222883e+07	1.081034e+07	8.109985e+07	4.070563e+07	4.216258e+07	4.038116e+07

Table C.14: Comparison of hypervolume indicator values for different optimizers on the WFG2 test problem.

Dim.	Stat.	NSGA-II	MOEA/D TCH	MOEA/D NTCH	MOEA/D PBI	SMS-EMOA	Δ_p-DDE	R2-MOGA	R2-MOGAw	R2-MODE	R2-IBEA	MOMBI TCH	MOMBI NTCH	MOMBI PBI
2D	med.	1.056834e+01	1.031620e+01	9.254194e+00	8.562011e+00	1.059374e+01	1.110396e+01	1.049714e+01	1.053975e+01	1.133364e+01	1.059186e+01	1.057496e+01	1.058443e+01	1.037753e+01
	avg.	1.059733e+01	9.879543e+00	9.822272e+00	9.038818e+00	1.058693e+01	1.110198e+01	1.048741e+01	1.053604e+01	1.132532e+01	1.058504e+01	1.055085e+01	1.062808e+01	1.028095e+01
	min.	1.049669e+01	8.973490e+00	8.802401e+00	7.919902e+00	1.038937e+01	1.103344e+01	9.186385e+00	1.040859e+01	1.122601e+01	1.046523e+01	9.241406e+00	1.051236e+01	9.796329e+00
	max.	1.137921e+01	1.056364e+01	1.059085e+01	1.035798e+01	1.061746e+01	1.117534e+01	1.130595e+01	1.129823e+01	1.136480e+01	1.060982e+01	1.137410e+01	1.143464e+01	1.121664e+01
	std.	1.555191e−01	6.595801e−01	6.758152e−01	7.352634e−01	2.935089e−02	2.935188e−02	1.569239e−01	8.551981e−02	2.838323e−02	2.477990e−02	2.047586e−01	1.974730e−01	2.523929e−01
3D	med.	9.676038e+01	8.348310e+01	7.438178e+01	7.930761e+01	1.007076e+02	9.724574e+01	9.773399e+01	9.848679e+01	9.807757e+01	9.976488e+01	9.726926e+01	9.861582e+01	9.680853e+01
	avg.	9.294482e+01	8.530760e+01	7.564650e+01	8.065022e+01	9.358952e+01	9.725256e+01	9.220152e+01	9.414204e+01	9.806217e+01	9.400347e+01	9.262112e+01	9.334010e+01	9.211069e+01
	min.	8.202849e+01	6.649721e+01	6.455183e+01	6.426446e+01	8.499928e+01	9.632287e+01	8.252900e+01	8.158626e+01	9.694848e+01	8.401966e+01	8.279672e+01	8.411528e+01	8.171586e+01
	max.	9.800972e+01	9.807069e+01	9.284340e+01	9.475782e+01	1.011362e+02	9.812582e+01	9.929144e+01	9.963081e+01	9.888000e+01	1.005083e+02	9.852800e+01	9.982419e+01	9.835445e+01
	std.	6.420325e+00	5.624846e+00	6.793015e+00	5.410482e+00	7.825166e+00	4.398087e−01	7.410519e+00	7.069748e+00	3.937220e−01	7.473167e+00	6.653694e+00	7.080228e+00	6.999452e+00
4D	med.	8.580452e+02	7.352646e+02	5.972418e+02	6.642789e+02	9.023297e+02	8.070014e+02	8.986689e+02	9.031349e+02	8.853085e+02	9.081908e+02	8.813632e+02	9.058905e+02	8.669970e+02
	avg.	8.367862e+02	7.407499e+02	6.228836e+02	6.718090e+02	8.442911e+02	8.056685e+02	8.376611e+02	8.530653e+02	8.854664e+02	8.637822e+02	8.446033e+02	8.692146e+02	8.311202e+02
	min.	7.143466e+02	5.159311e+02	5.189946e+02	5.345101e+02	7.381353e+02	7.647743e+02	7.298210e+02	7.393345e+02	8.748484e+02	7.514085e+02	7.484682e+02	7.516358e+02	7.089115e+02
	max.	8.803917e+02	8.964366e+02	8.455987e+02	8.243408e+02	9.237064e+02	8.492161e+02	9.194767e+02	9.167273e+02	8.940807e+02	9.185441e+02	9.159024e+02	9.222784e+02	8.909363e+02
	std.	5.197759e+01	5.399551e+01	6.834466e+01	4.417932e+01	7.890589e+01	1.364741e+01	7.752414e+01	7.435120e+01	3.291263e+00	7.080310e+01	6.477972e+01	6.591363e+01	6.475272e+01
5D	med.	8.929884e+03	7.829372e+03	6.044414e+03	6.910237e+03	9.699517e+03	8.234614e+03	9.922286e+03	9.883547e+03	9.727051e+03	9.927452e+03	9.945436e+03	1.004203e+04	9.240369e+03
	avg.	8.715990e+03	7.999154e+03	6.292882e+03	6.966263e+03	9.094139e+03	8.216906e+03	9.360747e+03	9.247260e+03	9.726322e+03	9.588148e+03	9.502540e+03	9.520055e+03	8.788251e+03
	min.	7.376337e+03	5.055141e+03	3.863645e+03	5.363118e+03	7.869579e+03	7.691002e+03	7.987850e+03	8.042248e+03	9.617339e+03	8.093220e+03	8.101632e+03	8.198840e+03	7.565369e+03
	max.	9.330339e+03	9.715336e+03	8.760137e+03	8.637415e+03	1.005313e+04	8.757164e+03	1.009657e+04	1.007388e+04	9.812766e+03	1.010269e+04	1.018583e+04	1.026931e+04	9.595323e+03
	std.	5.528894e+02	8.053158e+02	8.981644e+02	5.276409e+02	8.608010e+02	1.886424e+02	8.593869e+02	8.535358e+02	2.965877e+01	7.006414e+02	7.992169e+02	8.431929e+02	7.393262e+02
6D	med.	1.081027e+05	9.590283e+04	7.243951e+04	8.524195e+04	1.228404e+05	9.727521e+04	1.272516e+05	1.271174e+05	1.251269e+05	1.270968e+05	1.269257e+05	1.286095e+05	1.139028e+05
	avg.	1.074662e+05	9.703651e+04	7.408828e+04	8.645396e+04	1.171675e+05	9.741925e+04	1.190465e+05	1.208405e+05	1.251863e+05	1.211791e+05	1.196512e+05	1.225002e+05	1.091532e+05
	min.	8.905776e+04	6.894487e+04	5.163365e+04	7.116415e+04	9.896231e+04	9.098477e+04	1.011702e+05	1.027996e+05	1.241439e+05	1.033963e+05	1.037204e+05	1.042317e+05	9.506095e+04
	max.	1.149832e+05	1.185328e+05	9.915942e+04	1.087461e+05	1.284959e+05	1.043727e+05	1.302028e+05	1.309020e+05	1.266075e+05	1.298999e+05	1.322741e+05	1.324932e+05	1.231415e+05
	std.	4.732781e+03	8.149111e+03	9.445114e+03	6.322231e+03	1.071638e+04	3.179418e+03	1.191138e+04	1.074409e+04	4.614298e+02	1.015511e+04	1.112846e+04	1.074149e+04	9.773934e+03
7D	med.	1.537517e+06	1.360209e+06	1.015974e+06	1.230139e+06	1.817102e+06	1.447805e+06	1.537589e+06	1.803138e+06	1.858559e+06	1.821598e+06	1.550316e+06	1.572270e+06	1.544180e+06
	avg.	1.511360e+06	1.338831e+06	1.018542e+06	1.230795e+06	1.701311e+06	1.447141e+06	1.684223e+06	1.698826e+06	1.858580e+06	1.723082e+06	1.651073e+06	1.677069e+06	1.528445e+06
	min.	1.232113e+06	8.811574e+05	6.816297e+05	9.352757e+05	1.442884e+06	1.346795e+06	1.446620e+06	1.451882e+06	1.832253e+06	1.502653e+06	1.468409e+06	1.441492e+06	1.307295e+06
	max.	1.693106e+06	1.686997e+06	1.423734e+06	1.497698e+06	1.919818e+06	1.565410e+06	1.927773e+06	1.921562e+06	1.875611e+06	1.919701e+06	1.936536e+06	1.911159e+06	1.765878e+06
	std.	9.043842e+04	1.574924e+05	1.534324e+05	9.873037e+04	1.743276e+05	4.709328e+04	1.917199e+05	1.783423e+05	8.485134e+03	1.618875e+05	1.583995e+05	1.596484e+05	1.328284e+05
8D	med.	2.446881e+07	2.253557e+07	1.686780e+07	2.053183e+07	2.998301e+07	2.148810e+07	3.176784e+07	3.147606e+07	3.109770e+07	3.165479e+07	2.681464e+07	3.099733e+07	2.619871e+07
	avg.	2.434648e+07	2.222245e+07	1.680252e+07	2.068263e+07	2.837180e+07	2.154779e+07	2.977179e+07	2.967087e+07	3.110851e+07	3.008904e+07	2.865954e+07	2.959286e+07	2.566377e+07
	min.	2.025755e+07	1.539614e+07	1.131650e+07	1.575147e+07	2.386410e+07	2.027845e+07	2.522370e+07	2.472044e+07	3.079214e+07	2.585079e+07	2.504881e+07	2.463296e+07	2.255003e+07
	max.	2.633585e+07	2.786271e+07	2.321270e+07	2.486662e+07	3.142348e+07	2.278880e+07	3.269185e+07	3.279636e+07	3.144480e+07	3.265354e+07	3.262059e+07	3.258992e+07	2.906286e+07
	std.	1.085017e+06	2.291467e+06	1.895094e+06	1.250451e+06	2.732431e+06	5.455143e+05	3.083728e+06	2.857286e+06	1.391374e+05	2.649888e+06	2.730782e+06	2.678507e+06	2.148209e+06
9D	med.	4.391613e+08	4.233744e+08	3.156705e+08	3.819245e+08	5.593446e+08	3.685165e+08	6.075989e+08	6.025596e+08	5.592964e+08	6.078008e+08	5.911064e+08	5.965633e+08	5.210862e+08
	avg.	4.392415e+08	4.212582e+08	3.195349e+08	3.875035e+08	5.471798e+08	3.693754e+08	5.834038e+08	5.811032e+08	5.591493e+08	5.899363e+08	5.639176e+08	5.789271e+08	5.069613e+08
	min.	4.002431e+08	2.930129e+08	2.111415e+08	3.508651e+08	4.462704e+08	3.550719e+08	4.841421e+08	4.829563e+08	5.448589e+08	4.930676e+08	4.824572e+08	4.867659e+08	4.270662e+08
	max.	4.694057e+08	5.229989e+08	4.424426e+08	4.667665e+08	5.793282e+08	3.930594e+08	6.218282e+08	6.173709e+08	5.737261e+08	6.211857e+08	6.232942e+08	6.187652e+08	5.588184e+08
	std.	1.231910e+07	3.965819e+07	4.762269e+07	2.343985e+07	3.501682e+07	8.020867e+06	4.947679e+07	4.682784e+07	5.045957e+06	4.196664e+07	5.146583e+07	4.131862e+07	3.942438e+07
10D	med.	8.756451e+09	8.737562e+09	6.642250e+09	8.065695e+09	1.116544e+10	7.208815e+09	1.277258e+10	1.260315e+10	1.054056e+10	1.274692e+10	1.253637e+10	1.255996e+10	1.102896e+10
	avg.	8.781409e+09	8.843681e+09	6.871359e+09	8.204751e+09	1.102356e+10	7.219050e+09	1.237788e+10	1.216290e+10	1.053285e+10	1.250636e+10	1.217893e+10	1.220692e+10	1.080559e+10
	min.	7.718258e+09	7.342721e+09	4.344775e+09	7.471289e+09	8.957500e+09	6.855479e+09	1.008451e+10	1.000153e+10	1.023637e+10	1.031943e+10	9.958952e+09	1.022977e+10	9.211370e+09
	max.	9.320955e+09	1.094106e+10	9.153010e+09	1.024960e+10	1.177153e+10	7.791966e+09	1.298303e+10	1.290395e+10	1.081126e+10	1.300051e+10	1.301250e+10	1.303710e+10	1.168455e+10
	std.	1.986247e+08	6.889871e+08	1.116406e+09	5.906145e+08	5.690336e+08	1.454627e+08	9.445569e+08	9.682822e+08	1.256696e+08	7.374835e+08	9.042729e+08	8.689123e+08	6.336224e+08

Table C.15: Comparison of hypervolume indicator values for different optimizers on the WFG3 test problem.

Dim.	Stat.	NSGA-II	MOEA/D TCH	MOEA/D NTCH	MOEA/D PBI	SMS-EMOA	Δ_p-DDE	R2-MOGA	R2-MOGAw	R2-MODE	R2-IBEA	MOMBI TCH	MOMBI NTCH	MOMBI PBI
2D	med.	$1.080929e+01$	$1.082073e+01$	$1.084268e+01$	$1.062089e+01$	$1.091293e+01$	$1.088348e+01$	$1.075011e+01$	$1.083164e+01$	$1.073261e+01$	$1.088766e+01$	$1.088683e+01$	$1.089172e+01$	$1.081771e+01$
	avg.	$1.080367e+01$	$1.079727e+01$	$1.082819e+01$	$1.059923e+01$	$1.091125e+01$	$1.088337e+01$	$1.074028e+01$	$1.082056e+01$	$1.072604e+01$	$1.088460e+01$	$1.088513e+01$	$1.088926e+01$	$1.080348e+01$
	min.	$1.072798e+01$	$1.057135e+01$	$1.058798e+01$	$9.929915e+00$	$1.081503e+01$	$1.081213e+01$	$1.063483e+01$	$1.067673e+01$	$1.060252e+01$	$1.083557e+01$	$1.076554e+01$	$1.083578e+01$	$1.045460e+01$
	max.	$1.085037e+01$	$1.089914e+01$	$1.090870e+01$	$1.080497e+01$	$1.094099e+01$	$1.092627e+01$	$1.080885e+01$	$1.087582e+01$	$1.078188e+01$	$1.091535e+01$	$1.091715e+01$	$1.092592e+01$	$1.087375e+01$
	std.	$2.689809e-02$	$7.401296e-02$	$5.845298e-02$	$1.385856e-01$	$1.705812e-02$	$1.976086e-02$	$3.588083e-02$	$3.767611e-02$	$3.311820e-02$	$1.842918e-02$	$1.851391e-02$	$1.689314e-02$	$6.641551e-02$
3D	med.	$7.396661e+01$	$7.358340e+01$	$7.466708e+01$	$6.768157e+01$	$7.605552e+01$	$6.598147e+01$	$7.397549e+01$	$7.437005e+01$	$7.062335e+01$	$7.525225e+01$	$7.413735e+01$	$7.525643e+01$	$7.291207e+01$
	avg.	$7.393729e+01$	$7.347701e+01$	$7.446448e+01$	$6.783458e+01$	$7.602105e+01$	$6.590806e+01$	$7.395483e+01$	$7.434212e+01$	$7.064316e+01$	$7.524283e+01$	$7.414499e+01$	$7.522102e+01$	$7.281491e+01$
	min.	$7.326206e+01$	$7.064458e+01$	$7.156235e+01$	$6.270223e+01$	$7.537755e+01$	$6.314836e+01$	$7.303794e+01$	$7.284886e+01$	$6.970303e+01$	$7.445464e+01$	$7.297164e+01$	$7.457805e+01$	$6.998879e+01$
	max.	$7.445144e+01$	$7.465372e+01$	$7.539304e+01$	$7.239273e+01$	$7.623791e+01$	$6.756226e+01$	$7.473092e+01$	$7.545268e+01$	$7.162136e+01$	$7.564052e+01$	$7.475672e+01$	$7.563180e+01$	$7.355500e+01$
	std.	$2.521584e-01$	$8.284694e-01$	$6.703682e-01$	$2.170551e+00$	$1.509563e-01$	$7.506648e-01$	$3.662618e-01$	$4.832915e-01$	$4.138130e-01$	$2.183356e-01$	$2.908267e-01$	$2.169986e-01$	$6.180284e-01$
4D	med.	$6.042265e+02$	$5.654308e+02$	$6.205386e+02$	$5.319824e+02$	$4.114227e+02$	$5.181768e+02$	$6.382255e+02$	$6.406672e+02$	$5.582757e+02$	$6.252561e+02$	$5.710341e+02$	$6.352570e+02$	$5.998385e+02$
	avg.	$6.043845e+02$	$5.714357e+02$	$6.201350e+02$	$5.365397e+02$	$4.114308e+02$	$5.193379e+02$	$6.380942e+02$	$6.402401e+02$	$5.580652e+02$	$6.230327e+02$	$5.749988e+02$	$6.356407e+02$	$5.991985e+02$
	min.	$5.834848e+02$	$5.409257e+02$	$5.908843e+02$	$4.938397e+02$	$3.963935e+02$	$4.885361e+02$	$6.239606e+02$	$6.203449e+02$	$5.398772e+02$	$5.953855e+02$	$5.534377e+02$	$6.219857e+02$	$5.739971e+02$
	max.	$6.239929e+02$	$6.136477e+02$	$6.408495e+02$	$5.861216e+02$	$4.333750e+02$	$5.477353e+02$	$6.494884e+02$	$6.547134e+02$	$5.793513e+02$	$6.410710e+02$	$6.124131e+02$	$6.532811e+02$	$6.119626e+02$
	std.	$7.684829e+00$	$1.788037e+01$	$9.858316e+00$	$1.918334e+01$	$7.080513e+00$	$1.149648e+01$	$5.686265e+00$	$6.453276e+00$	$7.170581e+00$	$1.019094e+01$	$1.299633e+01$	$5.479134e+00$	$7.264621e+00$
5D	med.	$5.968174e+03$	$5.442422e+03$	$6.187873e+03$	$5.013715e+03$	$4.076214e+03$	$4.888444e+03$	$6.691556e+03$	$6.728691e+03$	$5.561652e+03$	$6.443108e+03$	$5.584336e+03$	$6.313007e+03$	$5.836843e+03$
	avg.	$5.975131e+03$	$5.444623e+03$	$6.169376e+03$	$5.033589e+03$	$4.080444e+03$	$4.876523e+03$	$6.694540e+03$	$6.723385e+03$	$5.571339e+03$	$6.422983e+03$	$5.565168e+03$	$6.314265e+03$	$5.849068e+03$
	min.	$5.626657e+03$	$5.090863e+03$	$5.690648e+03$	$4.714730e+03$	$3.892292e+03$	$4.336813e+03$	$6.505695e+03$	$6.394385e+03$	$5.394724e+03$	$6.001860e+03$	$5.202548e+03$	$6.176961e+03$	$5.606006e+03$
	max.	$6.285037e+03$	$5.903073e+03$	$6.556340e+03$	$5.503081e+03$	$4.335865e+03$	$5.270422e+03$	$6.911828e+03$	$6.925501e+03$	$5.819613e+03$	$6.707774e+03$	$5.795084e+03$	$6.551263e+03$	$6.161033e+03$
	std.	$1.316193e+02$	$1.462548e+02$	$1.579990e+02$	$1.771060e+02$	$9.123342e+01$	$1.980250e+02$	$8.428345e+01$	$1.022821e+02$	$8.753428e+01$	$1.478931e+02$	$9.583778e+01$	$7.104411e+01$	$1.218805e+02$
6D	med.	$6.947715e+04$	$6.118550e+04$	$7.409561e+04$	$5.550813e+04$	$5.110492e+04$	$5.690649e+04$	$8.165422e+04$	$8.285271e+04$	$6.666537e+04$	$8.472536e+04$	$6.503620e+04$	$7.791069e+04$	$6.785660e+04$
	avg.	$6.983225e+04$	$6.137359e+04$	$7.383787e+04$	$5.544411e+04$	$5.108420e+04$	$5.639551e+04$	$8.157367e+04$	$8.291366e+04$	$6.675284e+04$	$8.461369e+04$	$6.511463e+04$	$7.792368e+04$	$6.778879e+04$
	min.	$6.552391e+04$	$5.804626e+04$	$6.798345e+04$	$4.977190e+04$	$4.779721e+04$	$5.036483e+04$	$7.848407e+04$	$7.889210e+04$	$6.455816e+04$	$8.011002e+04$	$6.148506e+04$	$7.550737e+04$	$6.484860e+04$
	max.	$7.387094e+04$	$6.809099e+04$	$8.031128e+04$	$6.241685e+04$	$5.367779e+04$	$6.415447e+04$	$8.487010e+04$	$8.672255e+04$	$6.962332e+04$	$8.757251e+04$	$7.206598e+04$	$8.072205e+04$	$7.080093e+04$
	std.	$1.733108e+03$	$1.912985e+03$	$2.136428e+03$	$2.291702e+03$	$1.251632e+03$	$2.992801e+03$	$1.442275e+03$	$1.406138e+03$	$1.010859e+03$	$1.459085e+03$	$1.500636e+03$	$9.474970e+02$	$1.390665e+03$
7D	med.	$9.691010e+05$	$8.122205e+05$	$1.000734e+06$	$6.969254e+05$	$6.265543e+05$	$7.238729e+05$	$1.055682e+06$	$1.157163e+06$	$8.958505e+05$	$1.266649e+06$	$8.939361e+05$	$1.063537e+06$	$9.278825e+05$
	avg.	$9.713403e+05$	$8.145525e+05$	$1.005046e+06$	$6.996527e+05$	$6.264764e+05$	$7.234293e+05$	$1.059337e+06$	$1.154533e+06$	$8.991558e+05$	$1.266179e+06$	$8.933735e+05$	$1.061432e+06$	$9.040661e+05$
	min.	$9.119627e+05$	$7.538223e+05$	$9.292635e+05$	$5.933325e+05$	$5.684170e+05$	$6.883811e+05$	$9.988626e+05$	$1.035376e+06$	$8.635073e+05$	$1.236996e+06$	$8.240472e+05$	$9.999701e+05$	$7.244090e+05$
	max.	$1.052841e+06$	$9.011238e+05$	$1.111506e+06$	$7.843637e+05$	$6.777921e+05$	$7.558265e+05$	$1.123123e+06$	$1.217087e+06$	$9.430463e+05$	$1.289550e+06$	$9.464357e+05$	$1.135122e+06$	$9.788843e+05$
	std.	$2.895581e+04$	$2.892285e+04$	$3.084000e+04$	$4.503282e+04$	$2.346811e+04$	$1.388153e+04$	$2.696504e+04$	$3.132954e+04$	$1.840501e+04$	$1.120998e+04$	$2.476407e+04$	$2.160473e+04$	$6.439099e+04$
8D	med.	$1.573432e+07$	$1.349834e+07$	$1.666934e+07$	$9.277903e+06$	$1.239844e+07$	$1.209251e+07$	$1.757073e+07$	$1.882158e+07$	$1.491623e+07$	$2.135198e+07$	$1.496593e+07$	$1.767501e+07$	$1.186300e+07$
	avg.	$1.575243e+07$	$1.361386e+07$	$1.664953e+07$	$9.503001e+06$	$1.237393e+07$	$1.209596e+07$	$1.759909e+07$	$1.883578e+07$	$1.494786e+07$	$2.131521e+07$	$1.501481e+07$	$1.770375e+07$	$1.174053e+07$
	min.	$1.440350e+07$	$1.249661e+07$	$1.532744e+07$	$7.614940e+06$	$1.160241e+07$	$1.162767e+07$	$1.650301e+07$	$1.764949e+07$	$1.442826e+07$	$1.987108e+07$	$1.409654e+07$	$1.701365e+07$	$7.658894e+06$
	max.	$1.681993e+07$	$1.525898e+07$	$1.787993e+07$	$1.151703e+07$	$1.323281e+07$	$1.251061e+07$	$1.884562e+07$	$2.005961e+07$	$1.564438e+07$	$2.172340e+07$	$1.590545e+07$	$1.869919e+07$	$1.417287e+07$
	std.	$4.362249e+05$	$5.697359e+05$	$4.689376e+05$	$1.019703e+06$	$3.299865e+05$	$1.862970e+05$	$4.400014e+05$	$5.373520e+05$	$2.619275e+05$	$2.685813e+05$	$3.596242e+05$	$3.377660e+05$	$1.271330e+06$
9D	med.	$2.945376e+08$	$2.494244e+08$	$3.092004e+08$	$1.334286e+08$	$2.627673e+08$	$2.258205e+08$	$3.274848e+08$	$3.434477e+08$	$2.789685e+08$	$3.996065e+08$	$2.839948e+08$	$3.356152e+08$	$2.099746e+08$
	avg.	$2.951807e+08$	$2.500507e+08$	$3.089239e+08$	$1.424420e+08$	$2.629889e+08$	$2.255183e+08$	$3.278326e+08$	$3.433815e+08$	$2.792802e+08$	$3.982823e+08$	$2.845720e+08$	$3.355348e+08$	$2.064797e+08$
	min.	$2.802294e+08$	$2.300386e+08$	$2.843121e+08$	$1.227793e+08$	$2.539128e+08$	$2.157021e+08$	$3.096017e+08$	$3.212501e+08$	$2.706198e+08$	$3.769503e+08$	$2.613939e+08$	$3.167355e+08$	$1.239402e+08$
	max.	$3.182473e+08$	$2.760080e+08$	$3.346849e+08$	$1.952890e+08$	$2.722513e+08$	$2.359073e+08$	$3.482528e+08$	$3.657256e+08$	$2.889755e+08$	$4.050716e+08$	$3.087636e+08$	$3.503008e+08$	$2.516549e+08$
	std.	$7.066818e+06$	$1.034951e+07$	$9.743760e+06$	$1.864563e+07$	$4.183469e+06$	$4.008788e+06$	$8.001253e+06$	$9.617603e+06$	$3.753466e+06$	$4.545478e+06$	$8.801272e+06$	$6.275900e+06$	$2.008803e+07$
10D	med.	$6.143779e+09$	$5.263896e+09$	$6.410566e+09$	$2.425768e+09$	$7.722317e+09$	$4.676175e+09$	$6.654170e+09$	$6.944757e+09$	$5.765740e+09$	$8.147315e+09$	$5.945838e+09$	$7.021610e+09$	$3.909088e+09$
	avg.	$6.130052e+09$	$5.279773e+09$	$6.400860e+09$	$2.428676e+09$	$7.443058e+09$	$4.671773e+09$	$6.648185e+09$	$6.937659e+09$	$5.769412e+09$	$8.136016e+09$	$6.098895e+09$	$7.065997e+09$	$3.750224e+09$
	min.	$5.827035e+09$	$4.824468e+09$	$5.949479e+09$	$2.236274e+09$	$5.892548e+09$	$4.528799e+09$	$6.338627e+09$	$6.537097e+09$	$5.620850e+09$	$7.651215e+09$	$5.575701e+09$	$6.651744e+09$	$2.065892e+09$
	max.	$6.395603e+09$	$6.091023e+09$	$6.936544e+09$	$2.657650e+09$	$7.962092e+09$	$4.882896e+09$	$7.018640e+09$	$7.613115e+09$	$5.951529e+09$	$8.341356e+09$	$7.085086e+09$	$7.662851e+09$	$4.564315e+09$
	std.	$1.250371e+08$	$2.428456e+08$	$2.338107e+08$	$7.984825e+07$	$6.327740e+08$	$7.032866e+07$	$1.345885e+08$	$1.727941e+08$	$6.603764e+07$	$1.175515e+08$	$4.031995e+08$	$2.186951e+08$	$6.198866e+08$

Table C.16: Comparison of hypervolume indicator values for different optimizers on the WFG4 test problem.

Dim.	Stat.	NSGA-II	MOEA/D TCH	MOEA/D NTCH	MOEA/D PBI	SMS-EMOA	Δ_p-DDE	R2-MOGA	R2-MOGAw	R2-MODE	R2-IBEA	MOMBI TCH	MOMBI NTCH	MOMBI PBI
2D	med.	8.402261e + 00	8.607865e + 00	8.621991e + 00	8.435212e + 00	8.624041e + 00	8.217368e + 00	8.515415e + 00	8.569914e + 00	8.172909e + 00	8.639795e + 00	8.627164e + 00	8.626990e + 00	8.509979e + 00
	avg.	8.402740e + 00	8.600385e + 00	8.616291e + 00	8.431820e + 00	8.615366e + 00	8.216373e + 00	8.512404e + 00	8.563387e + 00	8.164967e + 00	8.628237e + 00	8.622771e + 00	8.623034e + 00	8.509093e + 00
	min.	8.203686e + 00	8.453109e + 00	8.521426e + 00	8.268336e + 00	8.488859e + 00	8.144535e + 00	8.427318e + 00	8.411010e + 00	8.054434e + 00	8.533454e + 00	8.564456e + 00	8.537375e + 00	8.420897e + 00
	max.	8.535149e + 00	8.641490e + 00	8.645153e + 00	8.516000e + 00	8.672459e + 00	8.291247e + 00	8.572662e + 00	8.608518e + 00	8.229426e + 00	8.671199e + 00	8.651225e + 00	8.650488e + 00	8.561596e + 00
	std.	7.233192e − 02	3.054213e − 02	2.226713e − 02	4.591062e − 02	3.747269e − 02	2.789951e − 02	3.140398e − 02	3.379753e − 02	3.368487e − 02	3.233533e − 02	1.664951e − 02	2.015437e − 02	2.641076e − 02
3D	med.	6.624712e + 01	7.371026e + 01	6.829832e + 01	7.255097e + 01	7.722033e + 01	7.020343e + 01	7.297241e + 01	7.323581e + 01	6.926202e + 01	7.602284e + 01	7.333068e + 01	7.421111e + 01	7.332515e + 01
	avg.	6.632662e + 01	7.375399e + 01	6.796094e + 01	7.253488e + 01	7.722187e + 01	7.019103e + 01	7.294510e + 01	7.320697e + 01	6.927881e + 01	7.601575e + 01	7.339666e + 01	7.421154e + 01	7.331585e + 01
	min.	6.285120e + 01	7.337854e + 01	5.064696e + 01	7.120644e + 01	7.702278e + 01	6.903319e + 01	7.158659e + 01	7.161580e + 01	6.801149e + 01	7.562070e + 01	7.298402e + 01	7.354928e + 01	7.218659e + 01
	max.	6.840041e + 01	7.454147e + 01	7.536008e + 01	7.355646e + 01	7.736688e + 01	7.098861e + 01	7.399205e + 01	7.416222e + 01	7.047653e + 01	7.639395e + 01	7.465349e + 01	7.478859e + 01	7.405878e + 01
	std.	1.049171e + 00	2.190785e − 01	3.962500e + 00	5.120868e − 01	8.649989e − 02	3.620598e − 01	5.026532e − 01	5.566442e − 01	5.010688e − 01	1.543570e − 01	2.893348e − 01	2.666076e − 01	2.888702e − 01
4D	med.	5.436381e + 02	6.566333e + 02	4.794876e + 02	6.959348e + 02	3.930478e + 02	6.613466e + 02	7.058745e + 02	7.123503e + 02	6.609910e + 02	7.348977e + 02	6.593933e + 02	6.749665e + 02	7.119989e + 02
	avg.	5.440889e + 02	6.578855e + 02	4.757090e + 02	6.970242e + 02	3.950572e + 02	6.583558e + 02	7.053418e + 02	7.109780e + 02	6.612472e + 02	7.348269e + 02	6.593987e + 02	6.753218e + 02	7.119021e + 02
	min.	5.106963e + 02	6.402526e + 02	3.511337e + 02	6.700559e + 02	3.833243e + 02	6.208600e + 02	6.853580e + 02	6.969828e + 02	6.419444e + 02	7.251673e + 02	6.344163e + 02	6.500308e + 02	6.980444e + 02
	max.	5.820597e + 02	6.878323e + 02	6.796938e + 02	7.209944e + 02	4.350171e + 02	6.842162e + 02	7.214957e + 02	7.261831e + 02	6.873816e + 02	7.450069e + 02	7.054941e + 02	6.931214e + 02	7.202057e + 02
	std.	1.456814e + 01	1.088798e + 01	7.502973e + 01	9.625091e + 00	8.116507e + 00	1.361310e + 01	7.716458e + 00	6.258527e + 00	7.355996e + 00	3.797056e + 00	1.477584e + 01	8.756623e + 00	4.381689e + 00
5D	med.	5.173619e + 03	7.014378e + 03	4.197752e + 03	7.183985e + 03	4.294043e + 03	5.785981e + 03	8.064670e + 03	8.118096e + 03	7.449736e + 03	8.225646e + 03	7.826277e + 03	7.611915e + 03	7.758123e + 03
	avg.	5.178641e + 03	6.995600e + 03	4.078744e + 03	7.160882e + 03	4.341308e + 03	5.770549e + 03	8.058394e + 03	8.112544e + 03	7.439987e + 03	8.222874e + 03	7.826969e + 03	7.628931e + 03	7.753989e + 03
	min.	4.752303e + 03	6.362235e + 03	3.247571e + 03	6.236177e + 03	4.220233e + 03	5.350865e + 03	7.702646e + 03	7.894278e + 03	7.227092e + 03	8.092201e + 03	7.095634e + 03	6.942698e + 03	7.375978e + 03
	max.	5.609634e + 03	7.713655e + 03	4.871508e + 03	7.643359e + 03	8.534824e + 03	6.086980e + 03	8.301093e + 03	8.295196e + 03	7.647066e + 03	8.358094e + 03	8.191277e + 03	8.124530e + 03	7.996644e + 03
	std.	1.946625e + 02	2.353202e + 02	4.699194e + 02	2.884082e + 02	4.258081e + 02	1.511573e + 02	1.004235e + 02	7.710411e + 01	7.713842e + 01	5.461149e + 01	2.072889e + 02	2.839131e + 02	1.026923e + 02
6D	med.	5.968287e + 04	6.813388e + 04	4.405267e + 04	8.272420e + 04	5.551979e + 04	6.039282e + 04	1.064885e + 05	1.069683e + 05	9.653418e + 04	1.054024e + 05	8.863893e + 04	8.522044e + 04	9.294528e + 04
	avg.	5.966841e + 04	6.832264e + 04	4.551140e + 04	8.253695e + 04	5.585827e + 04	6.081722e + 04	1.063186e + 05	1.067909e + 05	9.644725e + 04	1.053236e + 05	8.849589e + 04	8.531579e + 04	9.228950e + 04
	min.	5.505789e + 04	5.053486e + 04	3.716622e + 04	6.665281e + 04	5.411255e + 04	5.347425e + 04	1.023245e + 05	1.036746e + 05	9.310056e + 04	1.030777e + 05	7.167535e + 04	6.174908e + 04	4.450593e + 04
	max.	6.513949e + 04	8.785432e + 04	5.819591e + 04	9.788082e + 04	6.227946e + 04	6.949551e + 04	1.097182e + 05	1.093415e + 05	9.930925e + 04	1.077189e + 05	9.910423e + 04	9.868637e + 04	9.998659e + 04
	std.	2.170738e + 03	9.047219e + 03	5.748510e + 03	5.877294e + 03	1.237856e + 03	2.937652e + 03	1.536552e + 03	1.252760e + 03	1.295710e + 03	9.748776e + 02	5.016455e + 03	7.337352e + 03	6.288478e + 03
7D	med.	7.626824e + 05	7.039987e + 05	5.992648e + 05	1.049779e + 06	7.721466e + 05	7.837904e + 05	1.473203e + 06	1.575570e + 06	1.359873e + 06	1.552771e + 06	1.065118e + 06	9.982085e + 05	1.266178e + 06
	avg.	7.681575e + 05	7.133513e + 05	6.012694e + 05	1.046112e + 06	7.674721e + 05	7.849812e + 05	1.472897e + 06	1.573748e + 06	1.358055e + 06	1.550362e + 06	1.064619e + 06	9.946800e + 05	1.149893e + 06
	min.	6.935230e + 05	5.254554e + 05	4.970695e + 05	8.813301e + 05	7.007245e + 05	6.823138e + 05	1.387596e + 06	1.510828e + 06	1.289778e + 06	1.477144e + 06	9.048193e + 05	8.007528e + 05	6.625547e + 05
	max.	8.950067e + 05	9.484654e + 05	7.124490e + 05	1.425434e + 06	8.064882e + 05	8.866423e + 05	1.552191e + 06	1.615588e + 06	1.445082e + 06	1.595986e + 06	1.231470e + 06	1.136421e + 06	1.445869e + 06
	std.	4.043502e + 04	8.606977e + 04	5.994848e + 04	1.215214e + 05	2.026713e + 04	3.772201e + 04	3.212789e + 04	2.290530e + 04	3.743394e + 04	2.704500e + 04	6.082803e + 04	6.542364e + 04	2.760156e + 05
8D	med.	1.298781e + 07	1.082788e + 07	9.412132e + 06	1.678641e + 07	1.380735e + 07	1.384845e + 07	2.648057e + 07	2.696762e + 07	2.367411e + 07	2.669825e + 07	1.710310e + 07	1.629466e + 07	2.031159e + 07
	avg.	1.297446e + 07	1.076820e + 07	9.423581e + 06	1.688463e + 07	1.387526e + 07	1.380412e + 07	2.651485e + 07	2.686409e + 07	2.369309e + 07	2.669547e + 07	1.728115e + 07	1.630008e + 07	1.852220e + 07
	min.	1.124428e + 07	7.911378e + 06	6.791281e + 06	1.450647e + 07	1.345856e + 07	1.149445e + 07	2.473918e + 07	2.296455e + 07	2.260423e + 07	2.567236e + 07	1.488189e + 07	1.174214e + 07	1.091310e + 07
	max.	1.417993e + 07	1.378625e + 07	1.105532e + 07	2.147867e + 07	1.614675e + 07	1.540554e + 07	2.764875e + 07	2.774171e + 07	2.461509e + 07	2.761079e + 07	2.022809e + 07	1.866129e + 07	2.346539e + 07
	std.	5.548191e + 05	1.323678e + 06	9.424407e + 05	1.622176e + 06	3.185371e + 05	7.627866e + 05	4.627726e + 05	6.424710e + 05	3.967887e + 05	3.585358e + 05	9.690983e + 05	1.091205e + 06	4.572484e + 06
9D	med.	2.483190e + 08	1.881403e + 08	1.739954e + 08	2.892747e + 08	3.061929e + 08	2.568861e + 08	5.178957e + 08	4.994474e + 08	4.356138e + 08	4.936697e + 08	3.104687e + 08	3.016706e + 08	3.524195e + 08
	avg.	2.479821e + 08	1.864362e + 08	1.741570e + 08	3.035038e + 08	3.069210e + 08	2.588273e + 08	5.156018e + 08	4.972857e + 08	4.352006e + 08	4.936749e + 08	3.091686e + 08	2.998064e + 08	3.194467e + 08
	min.	2.289088e + 08	1.319176e + 08	1.463208e + 08	2.591960e + 08	2.928161e + 08	2.152132e + 08	4.812925e + 08	4.371322e + 08	4.099766e + 08	4.730888e + 08	2.390237e + 08	2.474819e + 08	2.007298e + 08
	max.	2.648017e + 08	2.504104e + 08	2.084345e + 08	3.818363e + 08	3.419635e + 08	3.039391e + 08	5.404442e + 08	5.250443e + 08	4.502305e + 08	5.080616e + 08	3.421019e + 08	3.657954e + 08	4.320006e + 08
	std.	9.024546e + 06	2.571084e + 07	1.559876e + 07	2.721270e + 07	6.733861e + 06	2.010524e + 07	1.046592e + 07	1.479988e + 07	7.114416e + 06	6.696077e + 06	1.730878e + 07	2.012126e + 07	8.346708e + 07
10D	med.	5.314407e + 09	3.597912e + 09	3.459685e + 09	6.030432e + 09	8.045404e + 09	4.971920e + 09	1.058143e + 10	9.641915e + 09	8.727140e + 09	9.936323e + 09	6.321575e + 09	6.016075e + 09	6.746440e + 09
	avg.	5.318515e + 09	3.553657e + 09	3.495181e + 09	6.243653e + 09	8.034174e + 09	5.128531e + 09	1.053549e + 10	9.641559e + 09	8.719371e + 09	9.935956e + 09	6.299927e + 09	5.981243e + 09	6.234957e + 09
	min.	4.896400e + 09	2.689995e + 09	2.771260e + 09	5.476586e + 09	7.713205e + 09	4.189190e + 09	9.516036e + 09	8.860524e + 09	8.417787e + 09	9.534824e + 09	5.027208e + 09	4.755706e + 09	3.958771e + 09
	max.	5.691618e + 09	4.876270e + 09	4.065683e + 09	8.114881e + 09	8.572660e + 09	6.531381e + 09	1.112250e + 10	1.044798e + 10	9.125954e + 09	1.026240e + 10	7.107583e + 09	7.018982e + 09	8.336053e + 09
	std.	1.520509e + 08	4.877549e + 08	2.486847e + 08	6.002466e + 08	1.762863e + 08	5.355756e + 08	3.338487e + 08	3.440622e + 08	1.336270e + 08	1.458795e + 08	4.159559e + 08	4.721691e + 08	1.549937e + 09

Table C.17: Comparison of hypervolume indicator values for different optimizers on the WFG5 test problem.

Dim.	Stat.	NSGA-II	MOEA/D TCH	MOEA/D NTCH	MOEA/D PBI	SMS-EMOA	Δ_p-DDE	R2-MOGA	R2-MOGAw	R2-MODE	R2-IBEA	MOMBI TCH	MOMBI NTCH	MOMBI PBI
2D	med.	8.043419e + 00	8.116876e + 00	8.122947e + 00	8.074498e + 00	8.129609e + 00	8.017967e + 00	8.090092e + 00	8.095556e + 00	8.041423e + 00	8.134081e + 00	8.126378e + 00	8.128996e + 00	8.095134e + 00
	avg.	8.039761e + 00	8.116617e + 00	8.121583e + 00	8.070394e + 00	8.129865e + 00	8.009467e + 00	8.090813e + 00	8.097865e + 00	8.040608e + 00	8.136253e + 00	8.131573e + 00	8.132780e + 00	8.099139e + 00
	min.	7.907299e + 00	8.081123e + 00	8.088496e + 00	7.999538e + 00	8.070608e + 00	7.913530e + 00	8.045824e + 00	8.053872e + 00	7.950874e + 00	8.122114e + 00	8.110351e + 00	8.106734e + 00	8.074029e + 00
	max.	8.091154e + 00	8.159794e + 00	8.146367e + 00	8.104901e + 00	8.166340e + 00	8.094995e + 00	8.143790e + 00	8.134816e + 00	8.096239e + 00	8.171018e + 00	8.195226e + 00	8.181808e + 00	8.141177e + 00
	std.	2.968168e − 02	1.027119e − 02	9.931583e − 03	1.957614e − 02	1.262214e − 02	3.895232e − 02	1.884348e − 02	1.615282e − 02	2.955215e − 02	1.035994e − 02	1.657966e − 02	1.372834e − 02	1.664093e − 02
3D	med.	6.598639e + 01	6.973968e + 01	6.495133e + 01	7.075213e + 01	7.399014e + 01	6.921137e + 01	7.023336e + 01	7.001682e + 01	6.923947e + 01	7.263338e + 01	6.972734e + 01	7.099176e + 01	7.082050e + 01
	avg.	6.605882e + 01	6.976597e + 01	6.506149e + 01	7.074854e + 01	7.380784e + 01	6.918929e + 01	7.019862e + 01	6.999604e + 01	6.923435e + 01	7.259704e + 01	6.973229e + 01	7.098923e + 01	7.081965e + 01
	min.	6.453793e + 01	6.936479e + 01	5.311561e + 01	7.008848e + 01	7.340703e + 01	6.732695e + 01	6.839176e + 01	6.776808e + 01	6.729063e + 01	7.191027e + 01	6.929005e + 01	7.028204e + 01	6.993921e + 01
	max.	6.761195e + 01	7.105328e + 01	7.206676e + 01	7.136500e + 01	7.415203e + 01	7.044328e + 01	7.141937e + 01	7.098520e + 01	7.042415e + 01	7.336819e + 01	7.031182e + 01	7.172502e + 01	7.179174e + 01
	std.	7.305982e − 01	3.002604e − 01	3.938233e + 00	2.606644e − 01	2.826087e − 01	5.348797e − 01	6.326657e − 01	5.296783e − 01	6.113510e − 01	3.060789e − 01	2.112301e − 01	2.868983e − 01	3.503949e − 01
4D	med.	5.514359e + 02	6.446846e + 02	5.964827e + 02	6.948729e + 02	3.752183e + 02	6.284322e + 02	6.852148e + 02	6.856321e + 02	6.499659e + 02	7.104471e + 02	6.324102e + 02	6.570696e + 02	6.896220e + 02
	avg.	5.510702e + 02	6.413575e + 02	5.914301e + 02	6.942713e + 02	3.810921e + 02	6.274234e + 02	6.841278e + 02	6.847188e + 02	6.509815e + 02	7.108075e + 02	6.331595e + 02	6.573064e + 02	6.896403e + 02
	min.	5.230625e + 02	6.088849e + 02	3.563687e + 02	6.647383e + 02	3.674253e + 02	5.949814e + 02	6.546677e + 02	6.723217e + 02	6.309218e + 02	7.051284e + 02	6.055725e + 02	6.371450e + 02	6.789653e + 02
	max.	5.761791e + 02	6.794789e + 02	6.775783e + 02	7.158465e + 02	4.400446e + 02	6.487402e + 02	7.013197e + 02	7.016604e + 02	6.697088e + 02	7.208938e + 02	6.631522e + 02	6.815279e + 02	6.973531e + 02
	std.	1.122574e + 01	1.641960e + 01	5.692961e + 01	9.502580e + 00	1.355341e + 01	1.130325e + 01	7.893271e + 00	5.308831e + 00	8.643595e + 00	3.044489e + 00	1.451067e + 01	9.302623e + 00	3.581828e + 00
5D	med.	5.000345e + 03	7.443855e + 03	6.332454e + 03	7.487174e + 03	4.322375e + 03	5.792766e + 03	7.841457e + 03	7.862925e + 03	7.058632e + 03	8.046322e + 03	7.294655e + 03	7.710059e + 03	7.515820e + 03
	avg.	5.017541e + 03	7.410630e + 03	6.025836e + 03	7.483201e + 03	5.914798e + 03	5.796672e + 03	7.844224e + 03	7.858257e + 03	7.056976e + 03	8.040320e + 03	7.263320e + 03	7.669490e + 03	7.522794e + 03
	min.	4.505780e + 03	6.686180e + 03	3.273020e + 03	7.064514e + 03	4.011339e + 03	5.418729e + 03	7.626487e + 03	7.707365e + 03	6.802798e + 03	7.896554e + 03	6.704001e + 03	6.867470e + 03	7.208881e + 03
	max.	5.381611e + 03	7.940270e + 03	7.436792e + 03	7.901973e + 03	8.430234e + 03	6.141593e + 03	8.055412e + 03	8.002696e + 03	7.295184e + 03	8.163834e + 03	7.836754e + 03	8.058533e + 03	7.927130e + 03
	std.	1.804037e + 02	2.797719e + 02	1.038416e + 03	1.703072e + 02	2.075138e + 03	1.343242e + 02	9.094609e + 01	6.357653e + 01	1.187889e + 02	4.804328e + 01	2.954439e + 02	1.960012e + 02	1.213560e + 02
6D	med.	5.486112e + 04	8.693052e + 04	5.810585e + 04	9.160808e + 04	5.326937e + 04	6.249133e + 04	1.037413e + 05	1.038268e + 05	8.825585e + 04	1.033310e + 05	9.405506e + 04	8.915694e + 04	9.342507e + 04
	avg.	5.489732e + 04	8.607212e + 04	6.098373e + 04	9.123307e + 04	5.384990e + 04	6.262522e + 04	1.038283e + 05	1.038731e + 05	8.826025e + 04	1.033612e + 05	9.420949e + 04	8.883596e + 04	9.336512e + 04
	min.	4.980051e + 04	6.873202e + 04	4.053197e + 04	8.125284e + 04	5.160031e + 04	5.421739e + 04	9.945873e + 04	1.012646e + 05	8.325084e + 04	1.014139e + 05	8.055685e + 04	7.041762e + 04	7.061912e + 04
	max.	6.108935e + 04	9.715025e + 04	8.954424e + 04	9.760232e + 04	6.949188e + 04	6.924790e + 04	1.083139e + 05	1.063309e + 05	9.448103e + 04	1.052635e + 05	1.040347e + 05	1.023627e + 05	1.023013e + 05
	std.	2.209785e + 03	5.772381e + 03	1.169935e + 04	3.356762e + 03	2.228292e + 03	3.156587e + 03	1.444462e + 03	1.018763e + 03	1.745391e + 03	7.506298e + 02	4.728157e + 03	5.377407e + 03	3.977566e + 03
7D	med.	7.185489e + 05	9.468577e + 05	6.354231e + 05	1.195745e + 06	7.470079e + 05	8.534405e + 05	1.488831e + 06	1.542226e + 06	1.272135e + 06	1.509402e + 06	1.231151e + 06	1.025460e + 06	1.335266e + 06
	avg.	7.206783e + 05	9.541403e + 05	6.540940e + 05	1.173492e + 06	7.496225e + 05	8.512788e + 05	1.492174e + 06	1.538578e + 06	1.276933e + 06	1.501400e + 06	1.244622e + 06	1.024896e + 06	1.312338e + 06
	min.	6.411052e + 05	7.076598e + 05	5.170595e + 05	8.050368e + 05	7.015238e + 05	7.259144e + 05	1.429273e + 06	1.474994e + 06	1.191836e + 06	1.403983e + 06	9.854822e + 05	8.500892e + 05	8.028283e + 05
	max.	8.018880e + 05	1.308244e + 06	8.685789e + 05	1.357848e + 06	8.522936e + 05	9.394218e + 05	1.567635e + 06	1.579821e + 06	1.374304e + 06	1.570955e + 06	1.502944e + 06	1.277734e + 06	1.456584e + 06
	std.	3.482546e + 04	1.195674e + 05	8.043351e + 04	1.203076e + 05	2.029143e + 04	3.868575e + 04	2.946807e + 04	1.974325e + 04	3.667736e + 04	3.513889e + 04	1.237238e + 05	7.432176e + 04	1.066999e + 05
8D	med.	1.159778e + 07	1.368180e + 07	1.010988e + 07	1.872563e + 07	1.351215e + 07	1.439280e + 07	2.645510e + 07	2.650334e + 07	1.993626e + 07	2.606022e + 07	1.750639e + 07	1.514216e + 07	2.050383e + 07
	avg.	1.162239e + 07	1.381954e + 07	1.022745e + 07	1.859733e + 07	1.406756e + 07	1.432979e + 07	2.640773e + 07	2.650940e + 07	1.995146e + 07	2.593360e + 07	1.775193e + 07	1.512703e + 07	1.864678e + 07
	min.	1.059010e + 07	1.056371e + 07	8.145420e + 06	1.369752e + 07	1.289171e + 07	1.284872e + 07	2.538709e + 07	2.583190e + 07	1.879140e + 07	2.475067e + 07	1.452988e + 07	1.215464e + 07	1.054172e + 07
	max.	1.257622e + 07	1.862453e + 07	1.318537e + 07	2.181391e + 07	2.833489e + 07	1.546386e + 07	2.752824e + 07	2.714512e + 07	2.112294e + 07	2.691596e + 07	2.285165e + 07	1.842073e + 07	2.313523e + 07
	std.	4.471479e + 05	1.463352e + 06	9.840168e + 05	1.723438e + 06	2.543940e + 06	6.049557e + 05	4.131680e + 05	2.944521e + 05	4.501463e + 05	5.163159e + 05	1.626468e + 06	1.338612e + 06	4.337768e + 06
9D	med.	2.174094e + 08	2.305922e + 08	1.802393e + 08	3.324307e + 08	3.068460e + 08	2.676542e + 08	5.103371e + 08	5.025082e + 08	3.549482e + 08	4.798160e + 08	2.913755e + 08	2.656694e + 08	2.055817e + 08
	avg.	2.177236e + 08	2.333834e + 08	1.847879e + 08	3.279235e + 08	3.220910e + 08	2.659915e + 08	5.106258e + 08	5.025733e + 08	3.555012e + 08	4.794096e + 08	2.939024e + 08	2.638298e + 08	2.745134e + 08
	min.	1.997491e + 08	1.854080e + 08	1.472944e + 08	2.592417e + 08	2.917401e + 08	2.349126e + 08	4.974271e + 08	4.868510e + 08	3.397012e + 08	4.529265e + 08	2.504742e + 08	1.916171e + 08	1.928997e + 08
	max.	2.404990e + 08	3.060646e + 08	2.328872e + 08	3.917096e + 08	5.261185e + 08	2.886417e + 08	5.225329e + 08	5.153205e + 08	3.740309e + 08	4.930127e + 08	3.630712e + 08	3.203371e + 08	4.420259e + 08
	std.	7.456366e + 06	2.781763e + 07	1.861265e + 07	3.137790e + 07	5.360345e + 07	1.243438e + 07	5.854970e + 06	5.315019e + 06	6.799860e + 06	7.344428e + 06	2.286232e + 07	2.605688e + 07	8.264773e + 07
10D	med.	4.584477e + 09	4.574586e + 09	3.596432e + 09	6.534129e + 09	9.654504e + 09	5.448771e + 09	1.068818e + 10	1.035402e + 10	7.137522e + 09	9.710447e + 09	5.747460e + 09	5.303109e + 09	4.070623e + 09
	avg.	4.572699e + 09	4.537539e + 09	3.616649e + 09	6.445685e + 09	9.194611e + 09	5.455152e + 09	1.066867e + 10	1.034157e + 10	7.137858e + 09	9.698056e + 09	5.663694e + 09	5.303984e + 09	4.923299e + 09
	min.	4.056620e + 09	3.546419e + 09	2.737534e + 09	5.396896e + 09	7.589661e + 09	4.650902e + 09	1.024830e + 10	1.007474e + 10	6.814237e + 09	9.366984e + 09	3.893538e + 09	3.471840e + 09	3.773702e + 09
	max.	4.848850e + 09	5.691787e + 09	4.509543e + 09	7.489958e + 09	1.028863e + 10	5.929412e + 09	1.098133e + 10	1.057486e + 10	7.448482e + 09	9.984666e + 09	6.824897e + 09	6.426266e + 09	7.849462e + 09
	std.	1.264785e + 08	5.034787e + 08	3.894553e + 08	6.089859e + 08	8.606275e + 08	2.369476e + 08	1.406703e + 08	1.205180e + 08	1.203708e + 08	1.288354e + 08	5.336725e + 08	5.571961e + 08	1.344173e + 09

Table C.18: Comparison of hypervolume indicator values for different optimizers on the WFG6 test problem.

Dim.	Stat.	NSGA-II	MOEA/D TCH	MOEA/D NTCH	MOEA/D PBI	SMS-EMOA	Δ_p-DDE	R2-MOGA	R2-MOGAw	R2-MODE	R2-IBEA	MOMBI TCH	MOMBI NTCH	MOMBI PBI
2D	med.	8.198399e+00	7.955993e+00	8.082958e+00	7.739815e+00	8.290557e+00	8.383870e+00	8.219880e+00	8.252860e+00	8.414499e+00	8.296825e+00	8.311365e+00	8.311444e+00	8.205897e+00
	avg.	8.197206e+00	7.971961e+00	8.047853e+00	7.747859e+00	8.267252e+00	8.375367e+00	8.213968e+00	8.241236e+00	8.411896e+00	8.291430e+00	8.306684e+00	8.313134e+00	8.204390e+00
	min.	7.922492e+00	7.344567e+00	7.419382e+00	7.214627e+00	7.829334e+00	8.131968e+00	7.848123e+00	7.855328e+00	8.249492e+00	7.945132e+00	8.026150e+00	8.104609e+00	7.934520e+00
	max.	8.365323e+00	8.406431e+00	8.418566e+00	8.234679e+00	8.416587e+00	8.556755e+00	8.368359e+00	8.398362e+00	8.501731e+00	8.460240e+00	8.483398e+00	8.452581e+00	8.355683e+00
	std.	6.707658e−02	2.132452e−01	2.065814e−01	2.097158e−01	9.423930e−02	7.334322e−02	7.654552e−02	9.204397e−02	5.186673e−02	6.856778e−02	7.285841e−02	6.007773e−02	7.214439e−02
3D	med.	6.533248e+01	7.075609e+01	6.178565e+01	7.037546e+01	7.438432e+01	6.820303e+01	7.061023e+01	7.082778e+01	7.027634e+01	7.344659e+01	7.055486e+01	7.177054e+01	7.064981e+01
	avg.	6.532154e+01	7.069896e+01	5.983697e+01	7.029907e+01	7.437958e+01	6.816260e+01	7.056013e+01	7.084200e+01	7.011261e+01	7.346859e+01	7.057881e+01	7.177643e+01	7.072292e+01
	min.	6.315394e+01	6.830847e+01	4.485882e+01	6.328253e+01	7.319357e+01	6.438016e+01	6.838058e+01	6.915319e+01	6.704986e+01	7.228541e+01	6.938603e+01	7.068194e+01	6.945001e+01
	max.	6.701344e+01	7.238491e+01	7.057151e+01	7.328161e+01	7.523925e+01	7.054648e+01	7.216472e+01	7.271915e+01	7.231757e+01	7.453073e+01	7.178119e+01	7.323433e+01	7.237079e+01
	std.	8.952243e−01	8.222284e−01	6.402817e+00	1.307385e+00	3.838560e−01	1.505403e+00	7.477918e−01	6.898669e−01	1.137004e+00	3.755876e−01	5.212065e−01	5.334335e−01	6.127926e−01
4D	med.	5.310829e+02	6.301578e+02	4.033159e+02	6.916158e+02	3.860228e+02	5.442857e+02	6.896138e+02	7.005123e+02	6.802800e+02	7.287785e+02	6.416354e+02	6.631739e+02	6.995114e+02
	avg.	5.306902e+02	6.321811e+02	4.309106e+02	6.756297e+02	3.928837e+02	5.562915e+02	6.898694e+02	7.006026e+02	6.621593e+02	7.284762e+02	6.409725e+02	6.636700e+02	6.999433e+02
	min.	4.983728e+02	6.084806e+02	2.907193e+02	4.779654e+02	3.751016e+02	4.265367e+02	6.678222e+02	6.834021e+02	5.415568e+02	7.154691e+02	6.130462e+02	6.406045e+02	6.841927e+02
	max.	5.607935e+02	6.569206e+02	6.188954e+02	7.212529e+02	5.170160e+02	6.617834e+02	7.090052e+02	7.131034e+02	6.996298e+02	7.381682e+02	6.686926e+02	6.868423e+02	7.159170e+02
	std.	1.468210e+01	9.595374e+00	8.287990e+01	4.327641e+01	2.356036e+01	7.421731e+01	9.370016e+00	6.364672e+00	3.597864e+01	3.721526e+00	1.144702e+01	1.092326e+01	6.205130e+00
5D	med.	4.629934e+03	6.947170e+03	3.444497e+03	6.083947e+03	8.463090e+03	5.036667e+03	7.873426e+03	8.108202e+03	7.776712e+03	8.314185e+03	7.512366e+03	8.113862e+03	7.742084e+03
	avg.	4.635416e+03	6.950653e+03	3.886520e+03	6.190272e+03	7.760370e+03	5.069824e+03	7.876071e+03	8.107614e+03	7.705731e+03	8.315038e+03	7.490810e+03	8.093530e+03	7.739955e+03
	min.	4.120170e+03	6.391129e+03	3.181937e+03	4.703992e+03	4.164542e+03	4.593307e+03	7.602497e+03	7.912308e+03	5.961302e+03	8.136684e+03	7.137780e+03	7.658902e+03	7.506620e+03
	max.	5.100349e+03	7.564287e+03	5.793984e+03	7.701776e+03	8.637264e+03	5.650397e+03	8.114454e+03	8.314738e+03	8.038419e+03	8.459291e+03	7.672111e+03	8.378182e+03	7.959509e+03
	std.	2.019826e+02	2.513995e+02	6.449523e+02	8.157317e+02	1.598301e+03	2.050387e+02	1.018171e+02	7.909874e+01	3.446156e+02	5.874370e+01	1.133328e+02	1.642679e+02	9.287244e+01
6D	med.	4.761073e+04	8.409319e+04	3.847487e+04	6.359488e+04	5.556630e+04	4.597019e+04	1.042823e+05	1.085614e+05	1.002484e+05	1.081247e+05	9.387435e+04	9.616293e+04	9.634051e+04
	avg.	4.750881e+04	8.403717e+04	4.118283e+04	6.770052e+04	6.781128e+04	4.591750e+04	1.042277e+05	1.084473e+05	1.004079e+05	1.081302e+05	9.476508e+04	9.638075e+04	9.561168e+04
	min.	4.158537e+04	7.283864e+04	3.316718e+04	5.723719e+04	5.366528e+04	3.948072e+04	9.975875e+04	1.052566e+05	9.559366e+04	1.059814e+05	8.743242e+04	8.632237e+04	8.639624e+04
	max.	5.316428e+04	9.211034e+04	5.885318e+04	8.692755e+04	1.135627e+05	5.205203e+04	1.078238e+05	1.109185e+05	1.056982e+05	1.100730e+05	1.046794e+05	1.072251e+05	1.018491e+05
	std.	2.218968e+03	3.702314e+03	5.465761e+03	9.363197e+03	2.279236e+04	2.744808e+03	1.397019e+03	1.063683e+03	2.109452e+03	8.361316e+02	4.452830e+03	4.040541e+03	2.861954e+03
7D	med.	5.758959e+05	9.257741e+05	5.157778e+05	8.174321e+05	7.651943e+05	5.631466e+05	1.460292e+06	1.618442e+06	1.515949e+06	1.636875e+06	1.142412e+06	1.096667e+06	1.273273e+06
	avg.	5.781532e+05	9.320311e+05	5.275256e+05	8.233028e+05	7.861047e+05	5.643614e+05	1.459999e+06	1.618315e+06	1.512933e+06	1.637251e+06	1.138200e+06	1.099457e+06	1.215035e+06
	min.	4.749569e+05	7.301262e+05	3.410576e+05	7.805534e+05	6.813088e+05	4.917092e+05	1.367939e+06	1.575810e+06	1.139877e+06	1.591222e+06	9.356240e+05	1.014342e+06	6.496085e+05
	max.	6.848574e+05	1.133770e+06	6.874671e+05	1.148869e+06	1.691054e+06	6.604783e+05	1.523406e+06	1.671660e+06	1.598693e+06	1.676982e+06	1.467280e+06	1.289128e+06	1.481108e+06
	std.	3.556166e+04	9.074537e+04	5.168059e+04	4.450323e+04	1.289444e+05	3.557264e+04	3.117700e+04	2.096478e+04	5.016319e+04	1.635700e+04	1.197600e+05	3.915788e+04	2.341787e+05
8D	med.	9.296541e+06	1.337458e+07	8.169218e+06	1.390621e+07	1.384185e+07	9.512056e+06	2.655225e+07	2.827464e+07	2.403116e+07	2.815551e+07	1.918951e+07	1.752966e+07	2.007328e+07
	avg.	9.324613e+06	1.337349e+07	8.341145e+06	1.397680e+07	1.464813e+07	9.449651e+06	2.649272e+07	2.825463e+07	2.409941e+07	2.813432e+07	1.885493e+07	1.761584e+07	1.833705e+07
	min.	7.916147e+06	1.068787e+07	5.966225e+06	1.308321e+07	1.342127e+07	7.587994e+06	2.539712e+07	2.738672e+07	2.229417e+07	2.677226e+07	1.485032e+07	1.579089e+07	1.104695e+07
	max.	1.078135e+07	1.777074e+07	1.258902e+07	1.938983e+07	2.811992e+07	1.085002e+07	2.732889e+07	2.882212e+07	2.603136e+07	2.891979e+07	2.338487e+07	2.180354e+07	2.373431e+07
	std.	4.865645e+05	1.532617e+06	1.046179e+06	7.087609e+05	3.038526e+06	6.836613e+05	4.688279e+05	2.634557e+05	7.489747e+05	3.208167e+05	1.932117e+06	8.711573e+05	4.701845e+06
9D	med.	1.797243e+08	2.277564e+08	1.522483e+08	2.616315e+08	3.200054e+08	1.823990e+08	5.245650e+08	5.395201e+08	3.944129e+08	5.127369e+08	3.579743e+08	2.992875e+08	3.463672e+08
	avg.	1.794065e+08	2.274561e+08	1.506790e+08	2.607860e+08	3.571368e+08	1.794530e+08	5.249169e+08	5.398921e+08	3.934499e+08	5.131126e+08	3.552834e+08	2.983975e+08	3.188133e+08
	min.	1.620398e+08	1.813144e+08	8.541120e+07	2.430622e+08	2.999552e+08	1.459607e+08	5.076285e+08	5.191176e+08	3.553208e+08	4.943923e+08	2.850316e+08	2.126582e+08	2.079309e+08
	max.	1.935352e+08	3.090712e+08	2.441470e+08	3.574412e+08	5.203575e+08	2.080311e+08	5.416467e+08	5.535201e+08	4.328732e+08	5.329118e+08	4.375812e+08	3.782036e+08	4.174641e+08
	std.	6.719862e+06	2.938846e+07	2.378363e+07	1.362119e+07	6.946682e+07	1.300977e+07	6.786004e+06	5.857796e+06	1.378885e+07	7.422836e+06	3.585370e+07	2.691810e+07	7.398673e+07
10D	med.	3.837978e+09	4.246700e+09	3.076697e+09	5.310315e+09	8.845401e+09	3.662930e+09	1.113458e+10	1.119161e+10	7.087531e+09	1.001971e+10	6.819535e+09	4.795547e+09	6.700709e+09
	avg.	3.832608e+09	4.364400e+09	3.053759e+09	5.320243e+09	8.888280e+09	3.630691e+09	1.113564e+10	1.118317e+10	7.093735e+09	1.001362e+10	6.766917e+09	4.875523e+09	6.397196e+09
	min.	3.528675e+09	3.571576e+09	1.329755e+09	4.987042e+09	8.010218e+09	2.861973e+09	1.077319e+10	1.056327e+10	6.586990e+09	9.576448e+09	4.923735e+09	3.417946e+09	4.297980e+09
	max.	4.129361e+09	5.663319e+09	4.248431e+09	6.444359e+09	1.001556e+10	4.201883e+09	1.148347e+10	1.147126e+10	7.537655e+09	1.045908e+10	8.176068e+09	6.441128e+09	8.108881e+09
	std.	1.304981e+08	5.610065e+08	5.054195e+08	2.094273e+08	4.208213e+08	3.024254e+08	1.317243e+08	1.608895e+08	2.160106e+08	1.681741e+08	6.598270e+08	7.228997e+08	1.200734e+09

Table C.19: Comparison of hypervolume indicator values for different optimizers on the WFG7 test problem.

Dim.	Stat.	NSGA-II	MOEA/D TCH	MOEA/D NTCH	MOEA/D PBI	SMS-EMOA	Δ_p-DDE	R2-MOGA	R2-MOGAw	R2-MODE	R2-IBEA	MOMBI TCH	MOMBI NTCH	MOMBI PBI
2D	med.	8.504703e+00	8.626862e+00	8.640761e+00	8.501081e+00	8.648595e+00	8.645529e+00	8.510212e+00	8.594195e+00	8.578840e+00	8.664916e+00	8.640381e+00	8.647797e+00	8.575772e+00
	avg.	8.496168e+00	8.607483e+00	8.635188e+00	8.479757e+00	8.647091e+00	8.636725e+00	8.510422e+00	8.563717e+00	8.571050e+00	8.662777e+00	8.636950e+00	8.647571e+00	8.574442e+00
	min.	8.360640e+00	8.311574e+00	8.447057e+00	8.223255e+00	8.605502e+00	8.566737e+00	8.418814e+00	8.095398e+00	8.446053e+00	8.636817e+00	8.537964e+00	8.627091e+00	8.504507e+00
	max.	8.571561e+00	8.655344e+00	8.664380e+00	8.564254e+00	8.674638e+00	8.663867e+00	8.591234e+00	8.642467e+00	8.595462e+00	8.676610e+00	8.664411e+00	8.664633e+00	8.611422e+00
	std.	4.441915e−02	6.058798e−02	2.784170e−02	7.629052e−02	1.648421e−02	2.322885e−02	3.310268e−02	9.550421e−02	2.695979e−02	8.636309e−03	1.829080e−02	8.378594e−03	1.806206e−02
3D	med.	6.848436e+01	7.377252e+01	5.806926e+01	7.158144e+01	7.758681e+01	7.433119e+01	7.290086e+01	7.345501e+01	7.290508e+01	7.666449e+01	7.377041e+01	7.494624e+01	7.404968e+01
	avg.	6.847171e+01	7.385688e+01	6.071312e+01	7.084538e+01	7.758593e+01	7.430486e+01	7.292731e+01	7.345815e+01	7.281803e+01	7.666302e+01	7.383202e+01	7.496039e+01	7.397895e+01
	min.	6.648365e+01	7.349459e+01	4.782078e+01	6.508053e+01	7.752450e+01	7.310984e+01	7.115957e+01	7.187388e+01	7.073237e+01	7.640184e+01	7.338281e+01	7.455641e+01	7.251642e+01
	max.	7.019949e+01	7.470177e+01	7.516605e+01	7.418575e+01	7.762396e+01	7.524816e+01	7.392816e+01	7.466967e+01	7.375654e+01	7.685646e+01	7.460577e+01	7.539181e+01	7.465891e+01
	std.	8.051748e−01	2.554404e−01	5.597093e+00	2.260749e+00	1.971970e−02	4.330730e−01	5.444382e−01	5.523644e−01	5.556086e−01	8.888271e−02	2.842080e−01	1.980329e−01	4.207457e−01
4D	med.	4.996780e+02	6.815890e+02	4.926428e+02	7.109104e+02	3.934979e+02	6.729799e+02	7.088413e+02	7.201050e+02	6.915485e+02	7.567138e+02	6.780766e+02	7.036777e+02	7.278043e+02
	avg.	4.950940e+02	6.806759e+02	4.793916e+02	7.079086e+02	3.933686e+02	6.695096e+02	7.093104e+02	7.194620e+02	6.915284e+02	7.566763e+02	6.797429e+02	7.041609e+02	7.266777e+02
	min.	4.585869e+02	6.454274e+02	3.529497e+02	6.168809e+02	3.845495e+02	6.211562e+02	6.915378e+02	6.961591e+02	6.748466e+02	7.491660e+02	6.534668e+02	6.845851e+02	7.111254e+02
	max.	5.295568e+02	7.097558e+02	6.445533e+02	7.364975e+02	4.140168e+02	6.947232e+02	7.275123e+02	7.336746e+02	7.138966e+02	7.616801e+02	7.154148e+02	7.285928e+02	7.376116e+02
	std.	1.661458e+01	1.242416e+01	8.774536e+01	1.979033e+01	4.102186e+00	1.548993e+01	7.818398e+00	7.290635e+00	7.570164e+00	2.170323e+00	1.221340e+01	8.774531e+00	5.766343e+00
5D	med.	4.581494e+03	7.613603e+03	3.518326e+03	6.824832e+03	4.389039e+03	5.261429e+03	8.147017e+03	8.280371e+03	7.709541e+03	8.619556e+03	7.692143e+03	8.299983e+03	7.779964e+03
	avg.	4.581974e+03	7.573569e+03	3.987697e+03	6.692119e+03	6.324375e+03	5.266323e+03	8.141599e+03	8.271861e+03	7.711053e+03	8.623928e+03	7.669825e+03	8.227213e+03	7.797698e+03
	min.	4.055973e+03	6.646088e+03	3.312491e+03	5.429259e+03	4.230460e+03	4.976617e+03	7.841723e+03	7.972437e+03	7.505675e+03	8.528252e+03	7.147984e+03	7.572698e+03	7.423948e+03
	max.	5.063850e+03	7.957100e+03	6.271653e+03	7.647823e+03	8.845006e+03	5.559716e+03	8.410013e+03	8.403445e+03	7.962784e+03	8.720661e+03	8.062887e+03	8.659450e+03	8.197369e+03
	std.	1.860412e+02	2.447673e+02	8.130933e+02	5.364329e+02	2.214465e+03	1.226506e+02	1.061709e+02	7.116472e+01	9.641079e+01	3.843964e+01	1.912596e+02	2.267532e+02	1.666955e+02
6D	med.	5.268889e+04	8.801274e+04	3.975091e+04	7.031332e+04	5.595178e+04	5.218829e+04	1.080754e+05	1.104212e+05	9.974094e+04	1.115394e+05	9.629348e+04	1.000582e+05	9.370454e+04
	avg.	5.273139e+04	8.779621e+04	4.176570e+04	7.248123e+04	5.603654e+04	5.246955e+04	1.081103e+05	1.103203e+05	9.968901e+04	1.114732e+05	9.645915e+04	9.997789e+04	9.001702e+04
	min.	4.768233e+04	6.975562e+04	3.481339e+04	6.354243e+04	5.497228e+04	4.477344e+04	1.053113e+05	1.081891e+05	9.667935e+04	1.098409e+05	8.434412e+04	8.891664e+04	4.498396e+04
	max.	5.675878e+04	9.880760e+04	6.705377e+04	8.938255e+04	5.890542e+04	6.042130e+04	1.115843e+05	1.121306e+05	1.031626e+05	1.132433e+05	1.054070e+05	1.078565e+05	1.015285e+05
	std.	1.829317e+03	5.267269e+03	6.412307e+03	6.570827e+03	6.164381e+02	3.070249e+03	1.410753e+03	8.600368e+02	1.491739e+03	7.191713e+02	4.542967e+03	3.251826e+03	1.230366e+04
7D	med.	6.588675e+05	9.412831e+05	5.275699e+05	8.689784e+05	7.718860e+05	6.954845e+05	1.497727e+06	1.628666e+06	1.415037e+06	1.632747e+06	1.108584e+06	1.154726e+06	6.749686e+05
	avg.	6.601036e+05	9.588548e+05	5.365159e+05	8.817116e+05	7.637689e+05	6.951539e+05	1.494538e+06	1.625337e+06	1.414173e+06	1.623872e+06	1.127245e+06	1.171606e+06	9.855901e+05
	min.	5.694477e+05	7.165991e+05	4.643722e+05	8.330884e+05	7.002583e+05	5.796203e+05	1.409066e+06	1.580052e+06	1.313644e+06	1.536628e+06	9.296996e+05	1.032891e+06	6.743840e+05
	max.	7.374033e+05	1.251383e+06	7.309912e+05	1.173028e+06	8.530466e+05	7.908863e+05	1.573716e+06	1.691777e+06	1.486610e+06	1.682867e+06	1.440584e+06	1.348358e+06	1.528107e+06
	std.	3.809771e+04	1.114411e+05	4.236611e+04	5.382654e+04	2.607919e+04	4.521402e+04	3.436046e+04	2.201532e+04	3.476070e+04	4.084226e+04	1.066234e+05	6.155011e+04	3.363489e+05
8D	med.	1.128588e+07	1.371264e+07	8.119357e+06	1.474089e+07	1.410866e+07	1.155439e+07	2.711783e+07	2.837560e+07	2.415682e+07	2.836729e+07	1.849510e+07	1.831931e+07	1.145757e+07
	avg.	1.126592e+07	1.366528e+07	8.472570e+06	1.479252e+07	1.412459e+07	1.149401e+07	2.707942e+07	2.835807e+07	2.413456e+07	2.828529e+07	1.891702e+07	1.832524e+07	1.435635e+07
	min.	9.501038e+06	9.492094e+06	7.028833e+06	1.376811e+07	1.377528e+07	9.527608e+06	2.576277e+07	2.757953e+07	2.297867e+07	2.718126e+07	1.281851e+07	1.459351e+07	1.144279e+07
	max.	1.259708e+07	1.845966e+07	1.192321e+07	2.065369e+07	1.497527e+07	1.307830e+07	2.799873e+07	2.929733e+07	2.556623e+07	2.912573e+07	2.356691e+07	2.225555e+07	2.496964e+07
	std.	5.358108e+05	1.524736e+06	1.101527e+06	8.215860e+05	1.433778e+05	7.254081e+05	4.732436e+05	2.898557e+05	4.850471e+05	4.975036e+05	2.087757e+06	1.101044e+06	4.468953e+06
9D	med.	2.192067e+08	2.372689e+08	1.444611e+08	2.753702e+08	3.127783e+08	2.103898e+08	5.389564e+08	5.448620e+08	4.386912e+08	5.325717e+08	3.438641e+08	3.242924e+08	2.172197e+08
	avg.	2.192494e+08	2.400425e+08	1.502446e+08	2.721544e+08	3.139217e+08	2.091964e+08	5.391399e+08	5.444462e+08	4.382711e+08	5.309585e+08	3.460286e+08	3.247308e+08	2.551916e+08
	min.	2.001196e+08	1.771660e+08	1.236465e+08	2.518592e+08	2.930513e+08	1.678184e+08	5.193023e+08	5.279954e+08	4.220771e+08	5.104017e+08	2.180577e+08	2.677608e+08	2.168127e+08
	max.	2.348297e+08	3.100721e+08	2.191585e+08	2.989193e+08	3.521590e+08	2.390311e+08	5.549639e+08	5.572454e+08	4.527455e+08	5.482985e+08	4.438543e+08	3.896908e+08	4.378971e+08
	std.	6.942984e+06	3.098686e+07	2.000005e+07	9.767556e+06	9.031780e+06	1.667939e+07	6.992072e+06	5.623291e+06	6.556562e+06	9.048002e+06	3.622318e+07	2.313491e+07	6.996237e+07
10D	med.	4.751688e+09	4.239049e+09	2.820428e+09	5.555775e+09	8.587573e+09	4.043786e+09	1.149293e+10	1.128382e+10	8.557179e+09	1.079123e+10	6.746903e+09	6.546829e+09	4.537529e+09
	avg.	4.744373e+09	4.416705e+09	2.990753e+09	5.591367e+09	9.384584e+09	4.065421e+09	1.147875e+10	1.126482e+10	8.568650e+09	1.079130e+10	6.778277e+09	6.592880e+09	4.956644e+09
	min.	4.400981e+09	3.370583e+09	2.441213e+09	5.150938e+09	7.680944e+09	3.453819e+09	1.106173e+10	1.077925e+10	8.227219e+09	1.051288e+10	5.475389e+09	5.683925e+09	4.514491e+09
	max.	5.076689e+09	5.984284e+09	4.523598e+09	7.524643e+09	1.102677e+10	4.832496e+09	1.176877e+10	1.153472e+10	9.014280e+09	1.116177e+10	8.176208e+09	7.680751e+09	8.289543e+09
	std.	1.297099e+08	5.895958e+08	4.621721e+08	3.194888e+08	1.191104e+09	3.155603e+08	1.375012e+08	1.379507e+08	1.384150e+08	1.457900e+08	4.789153e+08	4.010393e+08	9.882030e+08

Table C.20: Comparison of hypervolume indicator values for different optimizers on the WFG8 test problem.

Dim.	Stat.	NSGA-II	MOEA/D TCH	MOEA/D NTCH	MOEA/D PBI	SMS-EMOA	Δ_p-DDE	R2-MOGA	R2-MOGAw	R2-MODE	R2-IBEA	MOMBI TCH	MOMBI NTCH	MOMBI PBI
	med.	7.414018e+00	7.656952e+00	7.690919e+00	7.471072e+00	7.589346e+00	7.497658e+00	7.474729e+00	7.576929e+00	7.378723e+00	7.616577e+00	7.606740e+00	7.630038e+00	7.423013e+00
	avg.	7.410571e+00	7.639845e+00	7.694180e+00	7.452049e+00	7.582036e+00	7.498123e+00	7.476148e+00	7.573019e+00	7.379619e+00	7.613844e+00	7.604978e+00	7.626727e+00	7.413928e+00
2D	min.	7.249999e+00	7.423127e+00	7.509650e+00	7.156570e+00	7.403324e+00	7.440070e+00	7.406427e+00	7.497625e+00	7.329458e+00	7.496403e+00	7.533713e+00	7.567827e+00	7.311988e+00
	max.	7.559110e+00	7.758234e+00	7.785776e+00	7.584452e+00	7.655364e+00	7.555113e+00	7.538277e+00	7.648406e+00	7.432824e+00	7.694835e+00	7.662401e+00	7.674512e+00	7.498454e+00
	std.	5.872359e−02	6.728818e−02	4.521205e−02	9.989329e−02	4.636608e−02	2.254993e−02	2.800474e−02	3.551417e−02	2.282147e−02	3.352068e−02	2.634150e−02	2.348470e−02	4.488246e−02
	med.	6.136014e+01	6.820326e+01	5.646411e+01	6.708345e+01	7.097425e+01	6.397082e+01	6.531562e+01	6.742250e+01	6.411767e+01	6.975463e+01	6.728580e+01	6.812604e+01	6.730882e+01
	avg.	6.139574e+01	6.817618e+01	5.612454e+01	6.703574e+01	7.094368e+01	6.396532e+01	6.528575e+01	6.740995e+01	6.407731e+01	6.973631e+01	6.729550e+01	6.813741e+01	6.728380e+01
3D	min.	6.001858e+01	6.701647e+01	3.336945e+01	5.917705e+01	7.016162e+01	6.218915e+01	6.370779e+01	6.576586e+01	6.252444e+01	6.920995e+01	6.680781e+01	6.756006e+01	6.638516e+01
	max.	6.326637e+01	6.866650e+01	6.044412e+01	6.870213e+01	7.125347e+01	6.517403e+01	6.647188e+01	6.865250e+01	6.548347e+01	7.009516e+01	6.789230e+01	6.856292e+01	6.809308e+01
	std.	6.615193e−01	3.154008e−01	3.422397e+00	1.036261e+00	1.748198e−01	5.482895e−01	5.620513e−01	5.419310e−01	5.768989e−01	1.883877e−01	2.100657e−01	2.123212e−01	3.372698e−01
	med.	5.178129e+02	5.779936e+02	4.038885e+02	6.398205e+02	3.275459e+02	5.861289e+02	6.276188e+02	6.497390e+02	5.938900e+02	6.892077e+02	5.726299e+02	6.012084e+02	6.600620e+02
	avg.	5.171332e+02	5.777390e+02	4.004869e+02	6.340418e+02	3.279101e+02	5.862035e+02	6.256156e+02	6.505546e+02	5.927055e+02	6.891638e+02	5.777320e+02	6.028895e+02	6.598377e+02
4D	min.	4.820578e+02	5.625277e+02	3.085031e+02	4.223093e+02	3.157218e+02	5.605313e+02	5.920013e+02	6.256638e+02	5.639649e+02	6.800161e+02	5.567845e+02	5.844026e+02	6.458711e+02
	max.	5.377159e+02	5.912511e+02	5.519041e+02	6.738891e+02	3.480212e+02	6.073269e+02	6.446698e+02	6.652269e+02	6.138718e+02	6.962646e+02	6.072617e+02	6.335064e+02	6.782308e+02
	std.	1.139422e+01	6.103535e+00	4.017219e+01	3.256264e+01	6.447711e+00	9.694184e+00	9.742541e+00	7.022852e+00	9.604922e+00	2.966325e+00	1.362981e+01	1.035031e+01	5.742691e+00
	med.	4.701915e+03	5.570846e+03	3.778863e+03	5.442972e+03	3.509819e+03	4.897645e+03	7.083948e+03	7.352185e+03	6.497748e+03	7.782207e+03	5.774852e+03	5.863087e+03	7.288308e+03
	avg.	4.689386e+03	5.562041e+03	3.735984e+03	5.314447e+03	3.517666e+03	4.885363e+03	7.060591e+03	7.348311e+03	6.509441e+03	7.781706e+03	5.805813e+03	5.900438e+03	7.288489e+03
5D	min.	4.171368e+03	4.937473e+03	2.014974e+03	3.515850e+03	3.389561e+03	4.528120e+03	6.731320e+03	7.031697e+03	6.273247e+03	7.662703e+03	5.606126e+03	5.684910e+03	7.057470e+03
	max.	5.177757e+03	6.076709e+03	4.723501e+03	6.592384e+03	3.756003e+03	5.240091e+03	7.295738e+03	7.631910e+03	6.874916e+03	7.908423e+03	6.138275e+03	6.304071e+03	7.494259e+03
	std.	1.809007e+02	1.395862e+02	4.097390e+02	7.984237e+02	6.992227e+01	1.260683e+02	1.230738e+02	9.449962e+01	1.142037e+02	5.245949e+01	1.263648e+02	1.457249e+02	9.843935e+01
	med.	5.088712e+04	5.223261e+04	4.072945e+04	4.878435e+04	4.586166e+04	4.691687e+04	9.172090e+04	9.604253e+04	8.272414e+04	1.004391e+05	5.860406e+04	6.274318e+04	8.662297e+04
	avg.	5.105257e+04	5.308794e+04	4.056820e+04	4.829028e+04	4.586564e+04	4.715374e+04	9.174727e+04	9.595053e+04	8.277720e+04	1.005372e+05	6.013998e+04	6.253890e+04	8.592590e+04
6D	min.	4.621694e+04	4.731843e+04	2.964448e+04	2.854790e+04	4.374862e+04	4.083625e+04	8.719301e+04	9.265241e+04	7.785012e+04	9.840714e+04	5.630949e+04	5.076812e+04	6.125103e+04
	max.	5.696151e+04	6.545067e+04	5.419895e+04	6.934252e+04	4.891951e+04	5.534253e+04	9.745408e+04	9.936997e+04	8.580723e+04	1.035575e+05	6.848363e+04	7.391515e+04	9.065210e+04
	std.	2.081902e+03	4.135705e+03	4.568342e+03	1.143986e+04	8.763283e+02	3.497003e+03	1.881598e+03	1.474205e+03	1.581094e+03	9.739120e+02	3.038988e+03	4.384717e+03	3.543793e+03
	med.	6.295993e+05	5.429104e+05	5.197118e+05	5.179028e+05	5.959733e+05	5.992080e+05	1.231875e+06	1.390706e+06	1.175122e+06	1.492982e+06	6.888843e+05	7.093740e+05	1.181845e+06
	avg.	6.330144e+05	5.629083e+05	5.066791e+05	5.242716e+05	5.945667e+05	6.017049e+05	1.234433e+06	1.388300e+06	1.173963e+06	1.492344e+06	6.915676e+05	7.117015e+05	1.159868e+06
7D	min.	5.556352e+05	4.822746e+05	3.075501e+05	3.031279e+05	5.568856e+05	5.010612e+05	1.156930e+06	1.287122e+06	1.089088e+06	1.455914e+06	5.543968e+05	5.629852e+05	8.118912e+05
	max.	7.129799e+05	7.299513e+05	7.060748e+05	1.114366e+06	6.188820e+05	7.037062e+05	1.330749e+06	1.455884e+06	1.251766e+06	1.543029e+06	9.120445e+05	9.213501e+05	1.313261e+06
	std.	3.606480e+04	6.030944e+04	6.912706e+04	1.569529e+05	1.142818e+04	4.682738e+04	3.862599e+04	3.150896e+04	3.450404e+04	1.867656e+04	7.870499e+04	7.824725e+04	9.786009e+04
	med.	1.076408e+07	8.008727e+06	8.256383e+06	8.771063e+06	1.158736e+07	1.002094e+07	2.226608e+07	2.378661e+07	1.952435e+07	2.614665e+07	1.086252e+07	1.058468e+07	1.850981e+07
	avg.	1.067027e+07	8.322891e+06	8.109114e+06	9.097558e+06	1.158884e+07	1.000957e+07	2.228269e+07	2.375972e+07	1.953351e+07	2.612709e+07	1.108777e+07	1.056747e+07	1.822260e+07
8D	min.	9.635240e+06	6.839033e+06	4.298967e+06	4.360080e+06	1.093574e+07	8.223773e+06	2.045365e+07	2.248448e+07	1.838993e+07	2.535326e+07	8.404531e+06	8.213546e+06	6.880557e+06
	max.	1.148794e+07	1.150289e+07	1.233601e+07	1.999896e+07	1.214117e+07	1.170195e+07	2.345003e+07	2.507242e+07	2.086694e+07	2.691552e+07	1.470861e+07	1.571690e+07	2.099143e+07
	std.	4.409612e+05	1.034676e+06	1.260024e+06	3.002025e+06	2.616136e+05	8.119872e+05	5.680952e+05	4.634828e+05	5.105384e+05	3.253082e+05	1.433053e+06	1.636957e+06	1.764341e+06
	med.	2.038997e+08	1.367203e+08	1.495139e+08	1.512546e+08	2.768925e+08	1.805773e+08	4.359670e+08	4.481271e+08	3.549264e+08	4.894094e+08	1.986093e+08	1.848086e+08	3.205919e+08
	avg.	2.042755e+08	1.416472e+08	1.477315e+08	1.498197e+08	2.770711e+08	1.812214e+08	4.351253e+08	4.485837e+08	3.551497e+08	4.887379e+08	2.012299e+08	1.856252e+08	3.106811e+08
9D	min.	1.864115e+08	1.194040e+08	7.581260e+07	7.768961e+07	2.548909e+08	1.434406e+08	3.999600e+08	4.156698e+08	3.310152e+08	4.687732e+08	1.443829e+08	1.355544e+08	1.335061e+08
	max.	2.220711e+08	2.112974e+08	2.061202e+08	2.876264e+08	3.033815e+08	2.089691e+08	4.546783e+08	4.686136e+08	3.788361e+08	5.067261e+08	2.800170e+08	2.657768e+08	3.574663e+08
	std.	7.372729e+06	1.839811e+07	2.202065e+07	4.564036e+07	9.827313e+06	1.490355e+07	1.073683e+07	9.608111e+06	8.669565e+06	6.072263e+06	3.204641e+07	3.128124e+07	4.224189e+07
	med.	4.312222e+09	2.561497e+09	3.043866e+09	3.176005e+09	7.925282e+09	3.545467e+09	9.232464e+09	9.150706e+09	7.087128e+09	9.785364e+09	4.255596e+09	3.530811e+09	6.172762e+09
	avg.	4.321592e+09	2.707882e+09	3.026470e+09	3.208768e+09	7.765067e+09	3.514929e+09	9.191803e+09	9.169301e+09	7.079780e+09	9.787128e+09	4.169900e+09	3.456948e+09	5.905880e+09
10D	min.	3.967435e+09	2.098695e+09	1.951931e+09	1.472281e+09	6.692926e+09	2.756863e+09	8.686943e+09	8.840177e+09	6.701380e+09	9.511568e+09	2.544255e+09	2.538693e+09	1.701902e+09
	max.	4.709810e+09	3.898766e+09	4.279190e+09	7.665369e+09	8.802851e+09	4.231838e+09	9.636592e+09	9.549076e+09	7.366548e+09	1.005151e+10	5.770651e+09	5.503842e+09	6.987059e+09
	std.	1.550633e+08	4.121497e+08	3.710229e+08	1.156053e+09	6.802840e+08	3.094936e+08	2.138387e+08	1.450375e+08	1.522943e+08	1.179608e+08	7.443742e+08	6.545885e+08	9.424136e+08

Table C.21: Comparison of hypervolume indicator values for different optimizers on the WFG9 test problem.

Dim.	Stat.	NSGA-II	MOEA/D TCH	MOEA/D NTCH	MOEA/D PBI	SMS-EMOA	Δ_p-DDE	R2-MOGA	R2-MOGAw	R2-MODE	R2-IBEA	MOMBI TCH	MOMBI NTCH	MOMBI PBI
2D	med.	7.691902e+00	8.046242e+00	7.879194e+00	7.865473e+00	8.123898e+00	7.646212e+00	7.672756e+00	7.671188e+00	7.641120e+00	7.703374e+00	7.699809e+00	7.700877e+00	7.695820e+00
	avg.	7.875740e+00	7.973569e+00	7.891093e+00	7.838712e+00	8.011115e+00	7.639040e+00	7.870638e+00	7.879265e+00	7.641998e+00	7.950802e+00	7.893297e+00	7.919926e+00	7.944253e+00
	min.	7.668239e+00	7.571492e+00	7.022037e+00	7.219605e+00	7.684438e+00	7.519595e+00	7.620548e+00	7.638597e+00	7.606183e+00	7.677094e+00	7.680532e+00	7.690265e+00	7.678573e+00
	max.	8.287337e+00	8.280538e+00	8.327027e+00	8.162905e+00	8.385896e+00	7.695877e+00	8.338148e+00	8.347446e+00	7.674065e+00	8.383730e+00	8.439334e+00	8.369776e+00	8.286741e+00
	std.	2.602582e−01	2.076009e−01	2.346923e−01	2.130769e−01	3.070772e−01	3.338346e−02	2.910519e−01	2.901944e−01	1.454610e−02	3.060959e−01	2.938971e−01	3.035863e−01	2.761519e−01
3D	med.	6.527962e+01	6.534327e+01	5.836650e+01	6.516279e+01	7.369133e+01	6.668867e+01	6.605526e+01	6.602463e+01	6.566029e+01	7.254621e+01	6.527684e+01	6.653916e+01	6.696926e+01
	avg.	6.514404e+01	6.681222e+01	5.870186e+01	6.509685e+01	7.243494e+01	6.660841e+01	6.691457e+01	6.717851e+01	6.565937e+01	7.106331e+01	6.722452e+01	6.793165e+01	6.751350e+01
	min.	6.128491e+01	6.456790e+01	4.721122e+01	5.899720e+01	6.827014e+01	6.561104e+01	6.406720e+01	6.468427e+01	6.415069e+01	6.747216e+01	6.494306e+01	6.610249e+01	6.518482e+01
	max.	6.780847e+01	7.047450e+01	6.731869e+01	6.949716e+01	7.434864e+01	6.708276e+01	7.168178e+01	7.085422e+01	6.679318e+01	7.344112e+01	7.078491e+01	7.202515e+01	7.064832e+01
	std.	1.300269e+00	2.045506e+00	3.217780e+00	1.983030e+00	2.376786e+00	3.038930e−01	2.084607e+00	2.112440e+00	4.291575e−01	2.381627e+00	2.428549e+00	2.315506e+00	1.230659e+00
4D	med.	5.300733e+02	5.714567e+02	5.832796e+02	5.886619e+02	3.748838e+02	6.647851e+02	6.523326e+02	6.533806e+02	6.403066e+02	7.012552e+02	5.761338e+02	6.100624e+02	6.485794e+02
	avg.	5.413780e+02	5.765696e+02	5.338624e+02	5.739912e+02	3.788375e+02	6.646283e+02	6.556279e+02	6.551010e+02	6.396910e+02	6.961588e+02	5.820332e+02	6.111755e+02	6.460903e+02
	min.	4.582109e+02	5.471758e+02	3.303547e+02	3.908799e+02	3.417029e+02	6.492662e+02	6.241218e+02	6.358438e+02	6.193526e+02	6.791572e+02	5.673522e+02	6.009936e+02	5.991620e+02
	max.	6.128625e+02	6.203928e+02	6.380364e+02	6.536377e+02	4.608396e+02	6.753173e+02	6.961417e+02	6.818845e+02	6.555038e+02	7.100820e+02	6.280856e+02	6.460136e+02	6.665745e+02
	std.	4.120578e+01	1.351225e+01	9.340550e+01	5.748357e+01	2.273445e+01	4.549004e+00	1.516911e+01	9.342123e+00	7.609203e+00	1.015836e+01	1.412893e+01	7.949259e+00	1.486217e+01
5D	med.	5.764846e+03	5.865680e+03	4.786984e+03	6.458545e+03	4.140161e+03	6.913458e+03	7.512507e+03	7.465690e+03	7.204375e+03	7.879252e+03	6.453111e+03	6.164764e+03	6.697752e+03
	avg.	5.727189e+03	5.797702e+03	4.803194e+03	5.944315e+03	4.939073e+03	6.908978e+03	7.540150e+03	7.445514e+03	7.194991e+03	7.884235e+03	6.455636e+03	6.169278e+03	6.712606e+03
	min.	4.280141e+03	4.236082e+03	3.187078e+03	2.913353e+03	3.851469e+03	6.676350e+03	7.216084e+03	7.129465e+03	6.835440e+03	7.726729e+03	5.861417e+03	5.815316e+03	6.204190e+03
	max.	6.262670e+03	6.320721e+03	6.300747e+03	7.221621e+03	8.199542e+03	7.133778e+03	7.880198e+03	7.624332e+03	7.437330e+03	8.046273e+03	6.773108e+03	6.524031e+03	7.435088e+03
	std.	2.449390e+02	3.727852e+02	9.231974e+02	1.176378e+03	1.562664e+03	9.322844e+01	1.755420e+02	1.092773e+02	1.183286e+02	6.125363e+01	1.511765e+02	1.740056e+02	2.639866e+02
6D	med.	5.858313e+04	6.039915e+04	5.454309e+04	7.531459e+04	5.643296e+04	8.312766e+04	9.948890e+04	9.717952e+04	9.246754e+04	1.010312e+05	7.520660e+04	6.849397e+04	7.792511e+04
	avg.	5.817116e+04	6.166974e+04	5.180986e+04	6.812625e+04	6.254205e+04	8.292610e+04	9.931126e+04	9.656728e+04	9.227328e+04	1.010852e+05	7.592627e+04	6.996235e+04	7.728142e+04
	min.	4.867960e+04	4.250618e+04	3.429049e+04	2.752715e+04	5.058179e+04	5.126471e+04	9.512043e+04	9.034999e+04	8.521438e+04	9.732801e+04	5.612416e+04	5.640955e+04	6.897675e+04
	max.	6.716692e+04	7.688261e+04	6.910443e+04	9.004344e+04	1.060216e+05	8.970307e+04	1.031064e+05	1.013236e+05	9.557828e+04	1.042365e+05	8.989481e+04	8.492864e+04	8.486564e+04
	std.	3.928474e+03	8.064318e+03	1.003475e+04	1.794691e+04	1.582459e+04	4.780684e+03	1.649863e+03	2.487816e+03	2.074095e+03	1.479435e+03	6.866258e+03	6.460904e+03	3.846485e+03
7D	med.	6.323621e+05	7.053351e+05	6.762812e+05	7.587291e+05	8.051304e+05	1.088514e+06	1.420512e+06	1.341180e+06	1.310690e+06	1.456702e+06	8.101270e+05	6.741161e+05	9.671365e+05
	avg.	6.384818e+05	6.864183e+05	6.530240e+05	7.513025e+05	8.581351e+05	1.042663e+06	1.422545e+06	1.343000e+06	1.307359e+06	1.459950e+06	8.230703e+05	6.719463e+05	9.689602e+05
	min.	5.036688e+05	4.004194e+05	4.890448e+05	1.407016e+05	6.589928e+05	5.394781e+05	1.325909e+06	1.221795e+06	1.163620e+06	1.400465e+06	6.317589e+05	5.261187e+05	6.581308e+05
	max.	7.738779e+05	9.150218e+05	8.400844e+05	1.168134e+06	1.401082e+06	1.188216e+06	1.485754e+06	1.457133e+06	1.379770e+06	1.536552e+06	1.028823e+06	9.061386e+05	1.127031e+06
	std.	5.519328e+04	9.500074e+04	9.549866e+04	2.979684e+05	1.606389e+05	1.477581e+05	3.013138e+04	5.260217e+04	4.275769e+04	3.284949e+04	6.389270e+04	7.610115e+04	7.280027e+04
8D	med.	1.013170e+07	1.135647e+07	1.057083e+07	1.242834e+07	1.542715e+07	1.785157e+07	2.436908e+07	2.265258e+07	2.169063e+07	2.502594e+07	1.351279e+07	1.093869e+07	1.426530e+07
	avg.	1.026419e+07	1.088985e+07	1.026451e+07	1.101311e+07	1.671322e+07	1.747521e+07	2.428314e+07	2.282905e+07	2.145821e+07	2.506680e+07	1.357910e+07	1.096904e+07	1.418359e+07
	min.	8.361535e+06	5.089832e+06	6.169576e+06	1.840888e+06	1.291891e+07	8.584098e+06	2.094311e+07	2.068587e+07	1.844275e+07	2.389601e+07	7.661959e+06	8.517427e+06	8.519897e+06
	max.	1.211684e+07	1.455851e+07	1.374816e+07	1.875135e+07	2.668800e+07	1.941610e+07	2.542654e+07	2.517900e+07	2.287393e+07	2.615148e+07	1.779995e+07	1.391191e+07	1.623293e+07
	std.	7.222541e+05	1.947084e+06	1.490817e+06	4.752779e+06	3.590589e+06	1.785026e+06	6.283132e+05	1.052510e+06	8.783492e+05	4.563667e+05	1.574957e+06	1.263360e+06	1.192018e+06
9D	med.	1.916078e+08	2.128270e+08	1.862439e+08	2.042259e+08	3.385112e+08	2.713920e+08	4.569334e+08	4.209202e+08	3.850236e+08	4.601000e+08	2.468089e+08	2.041964e+08	2.415614e+08
	avg.	1.925167e+08	2.054503e+08	1.858569e+08	1.853978e+08	3.478025e+08	2.614818e+08	4.530753e+08	4.222975e+08	3.825263e+08	4.599595e+08	2.459670e+08	2.048265e+08	2.432177e+08
	min.	1.649565e+08	1.123482e+08	1.418733e+08	2.883366e+07	2.974242e+08	1.530858e+08	3.944506e+08	3.647748e+08	3.271847e+08	4.353291e+08	1.150329e+08	1.427793e+08	1.897724e+08
	max.	2.204062e+08	2.643534e+08	2.361205e+08	3.188717e+08	4.694565e+08	3.218445e+08	4.825099e+08	4.863847e+08	4.142138e+08	4.820497e+08	2.987134e+08	2.879048e+08	2.932679e+08
	std.	1.179596e+07	2.919996e+07	2.300492e+07	8.127499e+07	3.859259e+07	4.158010e+07	1.698714e+07	2.262651e+07	1.879681e+07	9.849063e+06	2.730898e+07	3.179065e+07	2.130520e+07
10D	med.	3.977346e+09	4.046560e+09	3.858338e+09	2.880690e+09	8.336035e+09	4.033182e+09	9.161036e+09	8.284437e+09	7.403559e+09	9.200227e+09	5.274879e+09	4.305756e+09	4.079922e+09
	avg.	3.975575e+09	3.856809e+09	3.786417e+09	2.920718e+09	8.250245e+09	4.016097e+09	9.173132e+09	8.392398e+09	7.423016e+09	9.184543e+09	5.290645e+09	4.182478e+09	4.101724e+09
	min.	3.256858e+09	1.908503e+09	2.798560e+09	8.316655e+08	7.338042e+09	3.039281e+09	8.093180e+09	7.590508e+09	6.573627e+09	8.658749e+09	2.061043e+09	2.755909e+09	1.803577e+09
	max.	4.674208e+09	5.271410e+09	4.510414e+09	6.008565e+09	9.096199e+09	5.906258e+09	1.000591e+10	9.724578e+09	8.173462e+09	9.693040e+09	6.922321e+09	6.210660e+09	5.261407e+09
	std.	2.687454e+08	8.365429e+08	4.180281e+08	1.394625e+09	4.985583e+08	5.731782e+08	3.322876e+08	5.050724e+08	3.812067e+08	2.052846e+08	6.758185e+08	7.653200e+08	4.588281e+08

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